A Brief Introduction to Copulas

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Introduction
Introduction

The Word *Copula* is a Latin noun that means "A link, tie, bond"

(Cassell's Latin Dictionary)
Introduction

• 1959: The word *Copula* appeared for the first time (Sklar 1959)
• 1981: The earliest paper relating copulas to the study of dependence among random variables (Schweizer and Wolff 1981)
• 1990's +: Academic literatures on how to use copulas in risk management.
Introduction

Why copula

- Non-linear dependence
- Be able to measure dependence for heavy tail distributions
- Very flexible: parametric, semi-parametric or non-parametric
- Be able to study asymptotic properties of dependence structures
- Computation is faster and stable with the two-stage estimation
- Can be more probabilistic or more statistical
- others...
**Introduction**

Why copula

Example: X \sim \text{lognormal}(0, 1) and Y \sim \text{lognormal}(0, \sigma^2)
Introduction

Marginal distribution functions

\[ F(x) = P[X \leq x], \quad G(y) = P[Y \leq y] \]

Joint distribution function

\[ H(x, y) = P[X \leq x, Y \leq y] \]

For each pair \((x, y)\), we can associate three numbers: \(F(x)\), \(G(y)\) and \(H(x, y)\)
Each pair of real number \((x, y)\) leads to a point of \((F(x), G(y))\) in the unit square \([0, 1][0, 1]\)
Introduction

The mapping, which assigns the value of the joint distribution function to each ordered pair of values of marginal distribution function is indeed a copula.
Introduction

Joint distribution function

\( G(y) \)

\( (x, y) \)

\( (0, 0) \)

\( F(x) \)

\( (1, 1) \)

Copulas

\( H(x, y) \)
Definition
Definition

A 2-dimensional copula is a distribution function on $[0, 1] \times [0, 1]$, with standard uniform marginal distributions.
Definition

If \((X, Y)\) is a pair of continuous random variables with distribution function \(H(x, y)\) and marginal distributions \(F_X(x)\) and \(F_Y(y)\) respectively, then \(U = F_X(x) \sim U(0, 1)\) and \(V = F_Y(y) \sim U(0, 1)\) and the distribution function of \((U, V)\) is a copula.

\[
C(u, v) = P(U \leq u, V \leq v) = P(X \leq F_X^{-1}(u), Y \leq F_Y^{-1}(v))
\]

\[
C(u, v) = H(F_X^{-1}(u), F_Y^{-1}(v))
\]
# Definition

\[ C : [0, 1]^2 \rightarrow [0, 1] \]

1. \[ C(u, 0) = C(0, v) = 0 \]

2. \[ C(u, 1) = u \quad C(1, v) = v \]

\[ (u_2, v_2) \]

\[ (u_1, v_1) \]

3. \[ v_1, v_2, u_1, u_2 \in [0, 1] ; u_2 \geq u_1, v_2 \geq v_1 \]
\[ C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) + C(u_1, v_1) \geq 0 \]

**Grounded**

**formal**

**2-Increasing**
Properties
Properties

Volume of Rectangle

\[ V_H = H(u_2, v_2) - H(u_1, v_2) - H(u_2, v_1) + H(u_1, v_1) \]
Properties

Copula is the C-Volume of rectangle $[0,u] \times [0,v]$

$$C(u, v) = V_c([0, u] \times [0, v])$$

Copula assigns a number to each rectangle in $[0,1] \times [0,1]$, which is nonnegative!
\[ C(u, v) = V_c([0, u] \times [0, v]) = V_c(A) + V_c(B) + V_c(C) + V_c(D) \]
**Properties**

**Example: Independent Copula**

\[ C(u_1, u_2) = u_1 \times u_2, \quad u \in [0,1]^2 \]
The graph of Independent Copula
Properties

Fréchet Lower bound Copula

\[ C_L(u_1, u_2) = \max \{ 0, u_1 + u_2 - 1 \}, u \in [0,1]^2 \]

Fréchet Upper bound Copula

\[ C_U(u_1, u_2) = \min \{ u_1, u_2 \}, u \in [0,1]^2 \]
Properties

Fréchet Lower bound Copula

Fréchet Upper bound Copula
Properties

Any copula will be bounded by Fréchet lower and upper bound copulas

\[
C_L(u_1, u_2) \leq C(u_1, u_2) \leq C_U(u_1, u_2), \forall u \in [0,1]^2
\]
Properties
Properties

Sklar's Theorem

Let $H$ be a joint df with marginal dfs $F$ and $G$, then there exists a copula $C$ such that

$$H(u, v) = C(F(u), G(v))$$

If $F$ and $G$ are continuous, then the copula is unique.
Properties

Important Consequences

- A copula describes how marginals are tied together
- A joint df can be decomposed into marginal dfs and copula
- Marginal dfs and copula can be studied separately (eg: MLE separately)
- Given a copula, we can get many multivariate distributions by selecting different marginal dfs
Properties

Other topics

- Survival copula
- Functional Invariance for monotone transform
- Non-parametric measures of dependence
- Tail dependence
- Simulation
Archimedean Copulas
Archimedean Copulas

\[ C(u, v) = g^{-1}(g(u) + g(v)) \]

\[ g : [0,1] \rightarrow [0,\infty] \quad g(1) = 0 \]

continuous, strictly decreasing convex function

\[ g^{-1}(t) = \begin{cases} g^{-1}(t), & 0 \leq t \leq g(0) \\ 0, & g(0) \leq t \leq \infty \end{cases} \]

pseudo-inverse of \( g \)
Archimedean Copulas

Archimedean Copula behaves like a binary operation

Commutative:

\[ C(u, v) = C(v, u), \quad \forall u, v \in [0, 1] \]

Associative:

\[ C(C(u, v), w) = C(u, C(v, w)), \quad \forall u, v, w \in [0, 1] \]

Order preserving:

\[ C(u_1, v_1) \leq C(u_2, v_2), \quad u_1 \leq u_2, v_1 \leq v_2, \in [0, 1] \]
Archimedean Copulas

Example:

Let \( g(t) = 1 - t , \quad t \in [0,1] \)

Then \( g^{[-1]}(t) = \max(1 - t , 0) \)

\[
C(u,v) = \max(u + v - 1,0)
\]

Fréchet Lower bound Copula is a kind of Archimedean Copula.
Archimedean Copulas

Archimedean Copulas have a wide range of applications for some reasons:

- Easy to be constructed
- Many families of copulas belong to it
- Many nice properties

Archimedean Copulas originally appeared in the study of probabilistic metric space, developing the probabilistic version of triangle inequality.
Constructing Copulas

- The Inverse Method
- Geometric Methods
Constructing Copulas

The Inverse Method

Given a bivariate distribution function $H$ with continuous margins $F$ and $G$, invert to obtain a copula:

$$C(u, v) = H(F^{-1}(u), G^{-1}(v))$$
Constructing Copulas

The Inverse Method (Example)

Gumbel's bivariate exponential distribution

\[ H_a(x, y) = \begin{cases} 1 - e^{-x} - e^{-y} + e^{-(x+y+axy)}, & x, y \geq 0 \\ 0, & \text{otherwise} \end{cases} \]

\[ F^{-1}(u) = -\ln(1-u) \]
\[ G^{-1}(v) = -\ln(1-v) \]

\[ C_a(u, v) = u + v - 1 + (1-u)(1-v)e^{-a\ln(1-u)\ln(1-v)} \]
Constructing Copulas

Geometric Methods

Without reference to distribution functions or random variables, we can obtain the copula via the C-Volume of rectangles in $[0, 1] \times [0, 1]$. 

let $C_a$ denote the copula with *support* as the line segments illustrated in the graph.
Constructing Copulas

Geometric Methods (Example)

Continuous

When $u \leq av$

$$C_a(u, v) = V_{C_a}([0, u] \times [0, 1]) = u$$
Constructing Copulas

Geometric Methods (Example)

continuous

When

\(1 - (1 - a)v > u > av\)

\[C_a(u, v) = C_a(av, v) = av\]
Constructing Copulas

Geometric Methods (Example)

continuous

When

\[ u > 1 - (1 - a) v \]

\[ V_{C_a}(A) = 0 \quad \Rightarrow \]

\[ C_a(u, v) = u + v - 1 \]
Constructing Copulas

Geometric Methods (Example)

continuous

\[
\begin{align*}
C_a(u, v) &= u & 0 \leq u \leq av & \leq a \\
C_a(u, v) &= av & 0 \leq av < u < 1 - (1-a)v \\
C_a(u, v) &= u + v - 1 & a \leq 1 - (1-a)v \leq u \leq 1
\end{align*}
\]

\(C_I: \) Fréchet Upper bound Copula

\(C_0: \) Fréchet Lower bound Copula
Reference
Reference


2. Nelsen, R.B. (1999), An Introduction to Copulas

THANK YOU!