THE UNIVERSITY OF BRITISH COLUMBIA

DEPARTMENT OF STATISTICS

TECHNICAL REPORT #227

An admissibility proof using an adaptive sequence of smoother proper priors approaching the target Improper prior

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January 2007

An admissibility proof using an adaptive sequence of smoother proper priors approaching the target improper prior

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Abstract

We give a sufficient condition for the admissibility of generalized Bayes estimators of the location vector of spherically symmetric distribution under squared error loss. Compared to the known results for the multivariate normal case, our sufficient condition is very tight and is close to being necessary. In particular we establish the admissibility of generalized Bayes estimators with respect to the harmonic prior and priors with slightly heavier tails than the harmonic prior. The key to our proof is an adaptive sequence of smooth proper priors approaching an improper prior fast enough to establish admissibility.

1 Introduction

Let $X = (X_1, \dots, X_p)'$ have a spherically symmetric density function $f(\|x-\theta\|)$ and consider estimation of a p-dimensional location parameter θ with a quadratic loss function $L(\theta, d) = (d-\theta)'(d-\theta) = \|d-\theta\|^2$. Therefore an estimator $\delta(X)$ is evaluated using the risk function

$$R(\theta, \delta) = E_\theta \left[\|\delta(X) - \theta\|^2 \right] = \int_{R^p} \|\delta(x) - \theta\|^2 f(\|x-\theta\|) dx.$$

An estimator δ is said to be admissible if no estimator δ' exists such that $R(\theta, \delta') \le R(\theta, \delta)$ for all θ with strict inequality for some θ. Hence admissibility is a desirable property for estimators. It is well-known that any proper Bayes estimator is admissible under very mild conditions. In many cases, however, a target estimator is generalized Bayes (gBayes), with respect to an improper prior like the Lebesgue measure. There is no guarantee that any gBayes estimator is admissible.

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A famous sufficient condition for admissibility of gBayes estimator has been given by Blyth (1951). A version of the Blyth result is the following. Let $g(\theta)$ be the target improper prior density and $g_1 \leq g_2 \leq \cdots \leq g$, an increasing sequence of proper prior densities approaching g. Each g_i is not necessarily normalized, so just satisfies $\int_\Theta g_i(\theta)d\theta < \infty$ for any fixed i. Let δ_g and δ_{gi} be the gBayes estimator with respect to $g(\theta)$ and the proper Bayes estimator with respect to $g_i(\theta)$, respectively. The non-standardized Bayes risk difference between δ_g and δ_{gi} with respect to $g_i(\theta)$ is given by

$$\Delta_i = \int_{R^p} \left[R(\theta, \delta_g) - R(\theta, \delta_{gi})\right] g_i(\theta)d\theta. \tag{1}$$

Blyth (1951) showed that if $\Delta_i \to 0$ as $i \to \infty$, δ_g is admissible. (See Theorem A.1 in Appendix.) Therefore a good choice of the sequence of proper priors approaching the target prior is the key to finding admissible gBayes estimators. As Berger (1985) pointed out, however, " Indeed, in general, very elaborate (and difficult to work with) choices of the g_i are needed." For example, when $p = 1$ under normality and spherical symmetry, Blyth (1951) and Stein (1959) showed that the most natural estimator X, which is gBayes with respect to $g(\theta) = 1$, is admissible by using a sequence of conjugate priors $g_i(\theta) = \exp(-\theta^2/i)$ and by using $g_i(\theta) = (1 + \theta^2/i)^{-1}$, respectively. These are relatively comprehensible choices. But when $p = 2$, neither a sequence $g_i(\theta) = \exp(-\|\theta\|^2/i)$ nor a sequence $g_i(\theta) = (1 + \theta^2/i)^{-1}$ works to show the admissibility of X under normality. Under spherically symmetry, James and Stein (1961) showed for $p = 2$ that $g_i(\theta) = h_i^2(\theta)$ works where

$$h_i(\theta) = \begin{cases} 1 & \|\theta\| \leq 1 \\ 1 - \frac{\log\|\theta\|}{\log i} & 1 \leq \|\theta\| \leq i/2 \\ \frac{\alpha(i, \|\theta\|)}{\|\theta\|\{\log\|\theta\|\}} & \|\theta\| > i/2 \end{cases}$$

and $\alpha(i, \|\theta\|)$ is chosen so that, for fixed θ, $\alpha(i, \|\theta\|)\|\theta\|^{-1}\{\log\|\theta\|\}^{-1} \to 1$ as $i \to \infty$ and $h_1 \leq h_2 \leq \cdots \leq 1$. On the other hand, X for $p \geq 3$, is inadmissible as shown by Stein (1956) under normality and Brown (1966) in quite a general setting. Therefore, in general, we would like to know the mechanisms for discrimination between admissibility and inadmissibility for gBayes estimators. In this paper, we will investigate a sufficient condition for the admissibility of gBayes estimators with respect to spherically symmetric priors.

Under normality, Brown (1971) gave a powerful condition for admissibility as follows. Let $G(\|\theta\|)$ be a spherically symmetric target prior density. Then the marginal density is also spherically symmetric as $m(\|x\|) = \int f(\|x-\theta\|)G(\|\theta\|)d\theta$. Brown (1971) showed that if the gBayes estimator with respect to $G(\|\theta\|)$ has a finite risk and

$$\int_1^\infty r^{1-p} m(r)^{-1} dr = \infty \tag{2}$$

then it is admissible. Since $m(r) \sim G(r)$ for large r under suitable, mild conditions as shown in Maruyama and Takemura (2006), the sufficient condition above reduces to

$$\int_1^\infty r^{1-p} G(r)^{-1} dr = \infty.$$

Brown also showed that if the integral in (2) is finite, the gBayes estimator is inadmissible. Needless to say, Brown (1971) dealt with quite general priors (which are permitted to have a non-differentiable density, to have some holes on R^p and not to be spherically symmetric) and gave a general sufficient condition for them. Unfortunately even if we assume that the target prior has a differentiable density and that the support is R^p in Brown's (1971) condition, we do not find an easier proof than Brown (1971) and his choice of the sequence is still extraordinarily complicated.

On the other hand, Brown and Hwang (1982) consider estimation of the natural mean vector of an exponential family under a quadratic loss function and so the intersection of their setting and ours is the normal case. Brown and Hwang (1982) give a sufficient condition for gBayes estimators to be admissible when the target prior density $g(\theta) = G(\|\theta\|)$ is differentiable. Their ingenuity lies in the decomposition of Δ_i given by (1), a result using the triangle and Cauchy-Schwartz inequalities, i.e.

$$\begin{aligned}\Delta_i &\le 8\int_{R^p} g(\theta)\|\nabla h_i(\theta)\|^2 d\theta + 2\int_{R^p} \left\| \frac{m(\nabla g|x)}{m(g|x)} - \frac{m(\nabla g h_i^2|x)}{m(gh_i^2|x)} \right\|^2 m(gh_i^2|x)dx \\ &= A_i + B_i\end{aligned}$$

where $g_i = gh_i^2$, $m(\psi|x) = \int_{R^p} \psi(\theta) f(\|\theta - x\|)d\theta$ and the gradient of a function $\rho(x)$ is denoted by

$$\nabla \rho(x) = (\frac{\partial}{\partial x_1}\rho(x), \ldots, \frac{\partial}{\partial x_p}\rho(x))'.$$

Hence the problem reduces to the simultaneous minimization problem of A_i and B_i. Brown and Hwang (1982) showed that when the sequence is chosen as

$$h_i(\theta) = \begin{cases} 1 & \|\theta\| \le 1 \\ 1 - \log\|\theta\|/\log i & 1 \le \|\theta\| \le i \\ 0 & \|\theta\| > i, \end{cases} \tag{3}$$

A_i and B_i go to 0 as $i \to \infty$ if

$$\int_1^\infty \frac{r^{p-3} G(r) dr}{\{\log(r+2)\}^2} < \infty \tag{4}$$

and

$$\int_{R^p} m \left(g \left\| \frac{\nabla g}{g} - \frac{m(\nabla g)}{m(g)} \right\|^2 \right) dx < \infty, \tag{5}$$

respectively. Needless to say, when A_i and B_i go to 0 as $i \to \infty$, the gBayes estimator is admissible by the Blyth method. They called (4) and (5) the "growth condition" and "asymptotic flatness condition", respectively. We see that their method of proof is much more transparent than Brown's (1971) and their sequence in (3) is simpler.

However the growth condition may be weaker than Brown's (1971) condition given by (2) e.g. $G(\|\theta\|) = \|\theta\|^{2-p} \log(\|\theta\| + 2)$, which is slightly heavier than $\|\theta\|^{2-p}$, satisfies (2), but not the growth condition. The reason seems to be that the sequence (3) does not depend on the target prior density g but is optimized for $g(\theta) \leq \|\theta\|^{2-p}$ for sufficiently large $\|\theta\|$. Furthermore if $G(\|\theta\|) \to \infty$ around the origin like $\|\theta\|^{2-p}$, it does not satisfy the asymptotic flatness condition. The reason found in their method of bounding B_i in order to apply the dominated convergence theorem, is very rough around the origin. Moreover the lack of power in Brown and Hwang (1982) stems from their choice of the sequence; h_i given in (3) is non-differentiable at $\|\theta\| = 1$ and truncated at $\|\theta\| = i$. When we deal with A_i, the truncated sequence does not cause trouble. But the truncated sequence generally does cause trouble when dealing with B_i.

In this paper, I naturally extend Brown-Hwang's decomposition method to the spherically symmetric case and give as strong a condition as that of Brown (1971) under normality. In Section 2, I consider a minimization problem, and show for the corresponding term A_i in Brown and Hwang (1982),

$$\inf_h \int_0^\infty \{h'(\eta)\}^2 \eta^{p-1} G(\eta) d\eta = 0$$

under some constraints. As an alternative to (3), I propose a smoother sequence for a solution of the problem

$$H_i(\eta) = \frac{\int_\eta^\infty e^{(\eta - r)/i} \beta(r) dr}{\int_\eta^\infty \beta(r) dr} \quad (i = 1, 2, \dots) \tag{6}$$

where

$$\beta(r) = -\frac{d}{dr}\left\{\left(\int_1^{2+r} \frac{s^{1-p}}{G(s)} ds\right)^{-1}\right\} = \frac{(r+2)^{1-p}/G(2+r)}{(\int_1^{2+r}\{s^{1-p}/G(s)\}ds)^2}, \tag{7}$$

which works very well when $\int_1^\infty \{s^{1-p}/G(s)\}ds = \infty$. This choice of the adaptive sequence to G is stimulated by the sequence in Zidek (1970), which was however truncated and non-differentiable. In Section 3, we show that our H_i also works well for proving that the corresponding B_i approaches 0 as $i \to \infty$ in spherically symmetric case. As a result, we can prove a strong sufficient condition for the admissibility of gBayes estimators by using an adaptive sequence of proper priors $G(\|\theta\|)H_i^2(\|\theta\|)$ which approaches the target improper prior $G(\|\theta\|)$. In particular, we show that the gBayes estimators with respect to the harmonic prior $G(\|\theta\|) = \|\theta\|^{2-p}$ and with respect to a prior with a slightly heavier tail

$$G(\|\theta\|) = \|\theta\|^{2-p} \log(\|\theta\| + c), \quad c > 1, \tag{8}$$

are admissible under mild regularity conditions on f.

Brown (1979) considered a more general problem than ours: estimation of θ for a general density $p(x-\theta)$ and a general loss function $W(\delta-\theta)$. He conjectured that the prior $g(\theta) \sim \|\theta\|^a$ with $a \le 2-p$ for sufficiently large $\|\theta\|$ leads to admissibility, regardless of the density p and the loss W. However there has been no exact results about admissibility in this type of setting unless normality and quadratic loss function are assumed. Hence our results support Brown's (1979) conjecture for the case of spherically symmetric family and a quadratic loss function.

The companion paper of Maruyama and Takemura (2006) deals with the same problem and gives a sufficient condition for admissibility without the assumption that the target prior is regularly varying. However the results in Maruyama and Takemura (2006) do not necessarily include the ones in this paper. Adaptive sequence of proper priors of the type suggested by Zidek (1970) as well as the assumption of the regularly varying prior yield more elegant results than in Maruyama and Takemura (2006).

2 A minimization problem

In this section, when a nonnegative function $k(\eta)$ satisfies

$$\int_0^1 k(\eta)d\eta < \infty \tag{9}$$

and

$$\int_1^\infty k(\eta)d\eta = \infty, \tag{10}$$

we consider a minimization problem

$$\inf_h \int_0^\infty \{h'(\eta)\}^2 k(\eta)d\eta = 0 \tag{11}$$

subject to

$$\int_0^\infty h^2(\eta)k(\eta)d\eta < \infty. \tag{12}$$

In Section 3, we set $k(\eta) = \eta^{p-1}G(\eta)$ where $G(\|\theta\|)$ is our target prior density. This type of minimization problem has been famous in mathematical physics. See Rukhin (1995) for the details. A very well-known sufficient condition on $k(\eta)$ to satisfy (11) is

$$\int_1^\infty \frac{d\eta}{k(\eta)} = \infty. \tag{13}$$

Indeed when (13) is satisfied, we define h_i $(i = 1, \dots)$ as

$$h_i(\eta) = \begin{cases} 1 & 0 < \eta < 1/2 \\ \frac{\int_\eta^i \{1/k(s)\}ds}{\int_{1/2}^i \{1/k(s)\}ds} & 1/2 \le \eta < i \\ 0 & \eta \ge i, \end{cases} \tag{14}$$

and easily find that

$$\int_0^\infty \{h_i'(\eta)\}^2 k(\eta)dt = \frac{1}{\int_{1/2}^i \{1/k(s)\}ds}, \tag{15}$$

which approaches 0 as $i \to \infty$. Since $h_i(\eta)$ is truncated at $\eta = i$, $\int_0^\infty h_i^2(\eta)k(\eta)d\eta < \infty$ is guaranteed.

In the statistical context, this type of sequence has been considered by Stein (1965), Zidek (1970) and Brown (1971). However, we will have to apply the same sequence for (11) to another minimization problem ($\inf B_i = 0$ as explained in Section 1). It is very hard to deal with a truncated and non-differentiable $h_i(\eta)$ like (14) in such a simultaneous minimization problem. Here we produce a differentiable and non-truncated sequence for our purpose.

We assume that $k(\eta)$ is continuously differentiable and regularly varying with index α, that is,

$$\lim_{\eta\to\infty} k(\eta x)/k(\eta) = x^{\alpha} \tag{16}$$

for any $x > 0$. When $k(\eta)$ satisfies (16), we sometimes use the notation $k(\eta) \in \mathrm{RV}_{\alpha}$. A typical $k(\eta)$ is $\eta^{\alpha}\{\log(\eta+2)\}^{\beta}$ for any β. Under (10) and (13), we have only to consider the case $-1 \le \alpha \le 1$.

We now define functions $H_i(\eta)$, $i = 1, 2, \ldots$ by

$$H_i(\eta) = \frac{\int_{\eta}^{\infty} e^{(\eta-r)/i}\beta(r)dr}{\int_{\eta}^{\infty}\beta(r)dr} \tag{17}$$

where

$$\beta(r) = -\frac{d}{dr}\left\{\left(\int_1^{2+r}\frac{1}{k(s)}ds\right)^{-1}\right\} = \frac{1/k(2+r)}{(\int_1^{2+r}\{1/k(s)\}ds)^2}. \tag{18}$$

The properties of β and H_i are given in the following theorems.

Theorem 2.1. *(I)* $\beta(r) \in \mathrm{RV}_{\alpha-2}$, $\int_r^{\infty}\beta(s)ds \in \mathrm{RV}_{\alpha-1}$ *and* $\beta'(r) \in \mathrm{RV}_{\alpha-3}$.

(II) $\lim_{r\to\infty} r\beta(r)/\int_r^{\infty}\beta(s)ds = \alpha - 1$ *and* $\lim_{r\to\infty} r\beta'(r)/\beta(r) = \alpha - 2$.

(III) $\beta(r)/\int_r^{\infty}\beta(s)ds$ *is bounded for* $r \ge 0$.

Proof. See Proposition 1.7 of Geluk and de Haan (1987) for part I and II.

By part I, $\beta(r)/\int_r^{\infty}\beta(s)ds \in \mathrm{RV}_{-1}$ and hence $\beta(r)/\int_r^{\infty}\beta(s)ds \to 0$ as $r \to \infty$. Since $\int_0^{\infty}\beta(r)dr < \infty$ by (18) and $k(r)$ is continuous, $\{\beta(r)/\int_r^{\infty}\beta(s)ds\}|_{r=0}$ is bounded. $\square$

By part II, there exists r_0 such that $\beta'(r) \le 0$ for all $r \ge r_0$. By redefining $\beta(r)$ as $\beta(r + r_0)$, we have $\beta(r)$ which is nondecreasing for $r > 0$.

Theorem 2.2. *(I)* $0 \le H_1(\eta) \le H_2(\eta) \le \cdots \le 1$. *For any fixed* η, $\lim_{i\to\infty} H_i(\eta) = 1$.

(II) For any fixed i, $\lim_{\eta\to\infty}\int_{\eta}^{\infty}\beta(r)dr\beta(\eta)^{-1}H_i(\eta) = i$.

(III) For any fixed η, $\lim_{i\to\infty} H_i'(\eta) = 0$.

(IV) $|H_i'(\eta)| < 2\beta(\eta)/\int_{\eta}^{\infty}\beta(r)dr$ *for all* $\eta > 0$.

(V) For any $\epsilon > 0$, there exists η_0 such that $-1-\epsilon < \eta H_i'(\eta)/H_i(\eta) \le 0$ for all $\eta \ge \eta_0$ and for all i.

Proof. It is obvious that $0 \le H_1(\eta) \le 1$ and $H_i(\eta)$ is increasing in i. For fixed η, $H_i(\eta) \uparrow 1$ by the monotone convergence theorem.

By integration by parts, the numerator of $H_i(\eta)$ is written as

$$\int_\eta^\infty e^{(\eta-r)/i}\beta(r)dr = i\beta(\eta) + i\int_\eta^\infty e^{(\eta-r)/i}\beta'(r)dr. \tag{19}$$

Therefore

$$H_i(\eta) = i\frac{\beta(\eta)}{\int_\eta^\infty \beta(r)dr} + i\frac{\int_\eta^\infty e^{(\eta-r)/i}\beta'(r)dr}{\int_\eta^\infty \beta(r)dr}. \tag{20}$$

(20) divided by (17) is

$$1 = i\frac{\beta(\eta)}{H_i(\eta)\int_\eta^\infty \beta(r)dr} + i\frac{\int_\eta^\infty e^{-r/i}\beta'(r)dr}{\int_\eta^\infty e^{-r/i}\beta(r)dr}.$$

Since $\beta'(r)/\beta(r) \to 0$ by part II of Theorem 2.1, the second term of the above equation for fixed i, converges to 0 as $\eta \to \infty$ by the L'Hospital theorem.

Using (19) again, differentiation of the numerator of $H_i(\eta)$ gives

$$\left(\int_\eta^\infty e^{(\eta-r)/i}\beta(r)dr\right)' = \frac{1}{i}\int_\eta^\infty e^{(\eta-r)/i}\beta(r)dr - \beta(\eta) = \int_\eta^\infty e^{(\eta-r)/i}\beta'(r)dr.$$

Therefore

$$H_i'(\eta) = \frac{\beta(\eta)\int_\eta^\infty e^{(\eta-r)/i}\beta(r)dr}{(\int_\eta^\infty \beta(r)dr)^2} - \frac{\int_\eta^\infty e^{(\eta-r)/i}\{-\beta'(r)\}dr}{\int_\eta^\infty \beta(r)dr}. \tag{21}$$

Note that $-\beta'(r) \ge 0$ by redefinition of β. Each term of the right hand side of (21) is nondecreasing in i and hence by the monotone convergence theorem

$$\lim_{i\to\infty} H_i'(\eta) = \frac{\beta(\eta)\int_\eta^\infty \beta(r)dr}{(\int_\eta^\infty \beta(r)dr)^2} - \frac{\int_\eta^\infty \{-\beta'(r)\}dr}{\int_\eta^\infty \beta(r)dr} = 0.$$

Furthermore we have

$$|H_i'(\eta)| \;<\; \left|\frac{\beta(\eta)\int_\eta^\infty e^{(\eta-r)/i}\beta(r)dr}{(\int_\eta^\infty \beta(r)dr)^2}\right| + \left|\frac{\int_\eta^\infty e^{(\eta-r)/i}\{-\beta'(r)\}dr}{\int_\eta^\infty \beta(r)dr}\right|$$

$$< \ 2\beta(\eta)/\int_\eta^\infty \beta(r)dr.$$

Dividing (21) by (17), we have

$$\eta\frac{H_i'(\eta)}{H_i(\eta)} = \eta\left(\frac{\beta(\eta)}{\int_\eta^\infty \beta(r)dr} + \frac{\int_\eta^\infty e^{-r/i}\beta'(r)dr}{\int_\eta^\infty e^{-r/i}\beta(r)dr}\right) \tag{22}$$

$$> \frac{\eta\beta(\eta)}{\int_\eta^\infty \beta(r)dr} - \frac{\int_\eta^\infty e^{-r/i}\{-r\beta'(r)/\beta(r)\}\beta(r)dr}{\int_\eta^\infty e^{-r/i}\beta(r)dr}$$

$$> \frac{\eta\beta(\eta)}{\int_\eta^\infty \beta(r)dr} - \sup_{r>\eta}\frac{-r\beta'(r)}{\beta(r)}.$$

By II of Theorem 2.1 the right hand side converges to -1. This implies that for any $\epsilon > 0$ there exists η_0 such that $\eta H_i'(\eta)/H_i(\eta) > -1-\epsilon$ for all $\eta \geq \eta_0$ and for all i. Finally we will prove that $H_i'(\eta) \leq 0$ for sufficiently large η independent of i. By II of Theorem 2.1,

$$\frac{\eta\int_\eta^\infty \beta(r)dr}{\beta(\eta)}\left(\frac{\beta(\eta)}{\int_\eta^\infty \beta(r)dr}\right)' = \eta\frac{\beta'(\eta)}{\beta(\eta)} + \eta\frac{\beta(\eta)}{\int_\eta^\infty \beta(r)dr} \to -1$$

and hence $\beta(\eta)/\int_\eta^\infty \beta(r)dr$ is eventually nonincreasing. Hence by redefining η_0 if necessary, we can assume that $\beta(\eta)/\int_\eta^\infty \beta(r)dr$ is monotone nonincreasing for $\eta \geq \eta_0$. By integration by parts on the numerator of the first term in (21), we have

$$\int_\eta^\infty e^{-r/i}\beta(r)dr = e^{-\eta/i}\int_\eta^\infty \beta(r)dr - i^{-1}\int_\eta^\infty e^{-r/i}\left\{\int_r^\infty \beta(s)ds\right\}dr$$

and hence

$$\left\{\frac{i\int_\eta^\infty \beta(r)dr}{\int_\eta^\infty e^{(\eta-r)/i}\left\{\int_r^\infty \beta(s)ds\right\}dr}\right\}H_i'(\eta)$$

$$= -\frac{\beta(\eta)}{\int_\eta^\infty \beta(r)dr} + \frac{\int_\eta^\infty e^{-r/i}\beta(r)dr}{\int_\eta^\infty e^{-r/i}\left\{\int_r^\infty \beta(s)ds\right\}dr}$$

$$= -\frac{\beta(\eta)}{\int_\eta^\infty \beta(r)dr} + \frac{\int_\eta^\infty \{\beta(r)/\int_r^\infty \beta(s)ds\}e^{-r/i}\left\{\int_r^\infty \beta(s)ds\right\}dr}{\int_\eta^\infty e^{-r/i}\left\{\int_r^\infty \beta(s)ds\right\}dr}$$

$$\leq -\frac{\beta(\eta)}{\int_\eta^\infty \beta(r)dr} + \sup_{t\geq\eta}\frac{\beta(t)}{\int_t^\infty \beta(r)dr},$$

which is zero for $\eta \geq \eta_0$. Hence we find that $H_i'(\eta) \leq 0$ for all $\eta \geq \eta_0$ and for all i.

□

Now we show that $H_i(\eta)$ works very well for the minimization problem (11).

Theorem 2.3. *Assume* $\int_1^\infty \{1/k(\eta)\}d\eta = \infty$. *Then* $H_i(\eta)$ *given by* (17) *and* (18) *satisfies*

$$\lim_{i\to\infty}\int_0^\infty \left\{\frac{d}{d\eta}H_i(\eta)\right\}^2 k(\eta)d\eta = 0 \tag{23}$$

and

$$\int_0^\infty H_i^2(\eta)k(\eta)d\eta < \infty \tag{24}$$

for fixed i.

Proof. Note that

$$\begin{aligned}
&\int_a^\infty \left\{\frac{\beta(\eta)}{\int_\eta^\infty \beta(r)dr}\right\}^2 k(\eta)d\eta \\
&= \int_a^\infty \frac{k(\eta)}{k(\eta+2)}\frac{d}{dr}\left\{-\left[\int_1^{2+r}\frac{1}{k(s)}ds\right]^{-1}\right\}\Bigg|_{r=\eta} d\eta \\
&\le \sup_{t\ge a}\frac{k(\eta)}{k(\eta+2)}\left[\int_1^{2+a}\frac{1}{k(s)}ds\right]^{-1} \\
&< \infty
\end{aligned}$$

for $a > 0$. Using part IV of Theorem 2.2 and part III of Theorem 2.1, we have

$$\begin{aligned}
&\int_0^\infty \left\{\frac{d}{d\eta}H_i(\eta)\right\}^2 k(\eta)d\eta \\
&\le 4\left(\int_0^1 + \int_1^\infty\right)\left\{\frac{\beta(\eta)}{\int_\eta^\infty \beta(r)dr}\right\}^2 k(\eta)d\eta \\
&\le 4\sup_{0\le t\le 1}\frac{\beta(\eta)}{\int_\eta^\infty \beta(r)dr}\int_0^1 k(\eta)d\eta + 4\int_1^\infty \left\{\frac{\beta(\eta)}{\int_\eta^\infty \beta(r)dr}\right\}^2 k(\eta)d\eta \\
&< \infty
\end{aligned}$$

which guarantees (23) by the dominated convergence theorem together with part III of Theorem 2.2.

For (24), since $H_i(\eta) \le i\beta(\eta)/\int_\eta^\infty \beta(s)ds$ for any $\eta > 0$ from (20), we have

$$\left(\int_0^{\eta_0} + \int_{\eta_0}^\infty\right) H_i^2(\eta)k(\eta)d\eta$$

$$
\begin{aligned}
&\leq \int_0^{\eta_0} k(\eta)d\eta + i^2 \int_{\eta_0}^{\infty} \left\{ \frac{\beta(\eta)}{\int_\eta^\infty \beta(r)dr} \right\}^2 k(\eta)d\eta \\
&< \infty,
\end{aligned}
$$

for any fixed i. □

3 Admissibility

In this section, we give a sufficient condition for admissibility of the gBayes estimator with respect to a spherically symmetric target prior density $g(\theta) = G(\|\theta\|)$. The assumptions on the behavior of G and f are following.

F1 There exist $r_0 > 0$, $L > 0$, and $s > 1$, such that $r^{p+s} f(r) \leq L$ for all $r \geq r_0$.

G1 $\eta G'(\eta)/G(\eta)$ is bounded for $0 < \eta < 1$.

G2 $\int_0^1 \eta^{p-1} G(\eta) d\eta < \infty$ and $\int_0^1 \eta^{p-1} |G'(\eta)| d\eta < \infty$.

G3 $\int_1^\infty \eta^{p-1} G(\eta) d\eta = \infty$.

G4 G is continuous differentiable and regularly varying.

FG1 $\int_0^\infty r^{p-1} f(r) G(r) dr < \infty$ and $\int_0^\infty r^{p-2} F(r) G(r) dr < \infty$.

From **G2** and **G3**, impropriety of G occurs at infinity. Notice that **G2** and **G3** correspond to the constraints (9) and (10) in Section 2.

The gBayes estimator δ_g with respect to the improper density $g(\theta)$ is written as

$$
\begin{aligned}
\delta_g(x) &= \frac{\int_{R^p} \theta f(\|x-\theta\|) g(\theta) d\theta}{\int_{R^p} f(\|x-\theta\|) g(\theta) d\theta} \\
&= x + \frac{\int_{R^p} (\theta - x) f(\|x-\theta\|) g(\theta) d\theta}{\int_{R^p} f(\|x-\theta\|) g(\theta) d\theta} \\
&= x + \frac{\int_{R^p} F(\|x-\theta\|) \nabla g(\theta) d\theta}{\int_{R^p} f(\|x-\theta\|) g(\theta) d\theta},
\end{aligned} \tag{25}
$$

which is well-defined if both $\int_{R^p} F(\|x-\theta\|) \nabla g(\theta) d\theta$ and $\int_{R^p} f(\|x-\theta\|) g(\theta) d\theta$ are integrable for all x. These are guaranteed by the assumptions above.

Write

$$
m(\psi|x) = \int_{R^p} \psi(\theta) f(\|\theta - x\|) d\theta
$$

$$M(\psi|x) = \frac{1}{C_f}\int_{R^p}\psi(\theta)F(\|\theta - x\|)d\theta$$

where $C_f = \{\pi^{p/2}/\Gamma(p/2+1)\}\int_0^\infty z^{p+1}f(z)dz$. Notice that $F(\cdot)/C_f$ is a probability density function because

$$\begin{aligned}
\int_{R^p}\|y-\theta\|^\alpha F(\|y-\theta\|)dy &= \int_{R^p}\|y\|^\alpha\{\int_{\|y\|} sf(s)ds\}dy \\
&= c_p\int_0^\infty r^{p-1+\alpha}\int_r^\infty sf(s)dsdr \\
&= c_p\int_0^\infty r^{p+1+\alpha}\int_1^\infty tf(rt)dtdr \\
&= c_p\int_1^\infty t\{\int_0^\infty r^{p+1+\alpha}f(rt)dr\}dt \\
&= c_p\int_1^\infty t^{-p-1-\alpha}dt\cdot\int_0^\infty z^{p+1+\alpha}f(z)dz \\
&= \frac{c_p}{p+\alpha}\int_0^\infty z^{p+1+\alpha}f(z)dz.
\end{aligned}$$

Then δ_g is written as

$$\delta_g(x) = x + C_f\frac{M(\nabla g|x)}{m(g|x)}.$$

Now we state the main theorem of this paper.

Theorem 3.1. *Assume **F1** with $s > 5$, **G1**–**G4** and **FG1**. Then the gBayes estimator with respect to $G(\|\theta\|)$ is admissible if $\int_1^\infty r^{1-p}\{G(r)\}^{-1}dr = \infty$.*

Proof of Theorem 3.1. Let δ_{gi} denote the Bayes estimator with respect to the proper prior density $g(\theta)h_i^2(\theta) = G(\|\theta\|)H_i^2(\|\theta\|)$ where $H_i(\eta)$ has been given by (17) and let $k(\eta)$ in (18) be $\eta^{p-1}G(\eta)$. Then the Bayes risk difference of δ_g and δ_{gi} with respect to the density $g(\theta)h_i^2(\theta)$ is written as

$$\begin{aligned}
\Delta_i &= \int_{R^p}\left[R(\theta,\delta_g) - R(\theta,\delta_{gi})\right]g(\theta)h_i^2(\theta)d\theta \\
&= \int_{R^p}\int_{R^p}[\|\delta_g-\theta\|^2 - \|\delta_{gi}-\theta\|^2]f(\|x-\theta\|)g(\theta)h_i^2(\theta)d\theta dx \\
&= \int_{R^p}\Bigg\{[\|\delta_g\|^2 - \|\delta_{gi}\|^2]\int_{R^p}f(\|x-\theta\|)g(\theta)h_i^2(\theta)d\theta \\
&\quad - 2(\delta_g-\delta_{gi})'\int_{R^p}\theta f(\|x-\theta\|)g(\theta)h_i^2(\theta)d\theta\Bigg\}dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_{R^p} \|\delta_g - \delta_{gi}\|^2 \left\{ \int_{R^p} f(\|x-\theta\|) g(\theta) h_i^2(\theta) d\theta \right\} dx \\
&= C_f^2 \int_{R^p} \left\| \frac{M(\nabla g|x)}{m(g|x)} - \frac{M(\nabla\{gh_i^2\}|x)}{m(gh_i^2|x)} \right\|^2 m(gh_i^2|x) dx \\
&= C_f^2 \int_{R^p} \left\| \frac{M(\nabla g|x)}{m(g|x)} - \frac{M(\nabla g h_i^2|x)}{m(gh_i^2|x)} - \frac{M(g\nabla h_i^2|x)}{m(gh_i^2|x)} \right\|^2 m(gh_i^2|x) dx.
\end{aligned}
$$

As in Brown and Hwang (1982), we have

$$
\begin{aligned}
\Delta_i \le\ & 2C_f^2 \int_{R^p} \left\| \frac{M(g\nabla h_i^2|x)}{m(gh_i^2|x)} \right\|^2 m(gh_i^2|x) dx \\
& + 2C_f^2 \int_{R^p} \left\| \frac{M(\nabla g|x)}{m(g|x)} - \frac{M(\nabla g h_i^2|x)}{m(gh_i^2|x)} \right\|^2 m(gh_i^2|x) dx \\
&= 2C_f^2 (A_i + B_i) \qquad \text{(say)}.
\end{aligned}
$$

Using the Cauchy-Schwartz inequality for A_i, we have

$$
\begin{aligned}
A_i &= 4 \int_{R^p} \|M(gh_i \nabla h_i|x)\|^2 \{m(gh_i^2|x)\}^{-1} dx \\
&\le 4 \int_{R^p} \frac{M(gh_i^2|x)}{m(gh_i^2|x)} M(g\|\nabla h_i\|^2|x) dx.
\end{aligned}
$$

By Theorem A.2 in Appendix, there exists L_1 such that

$$
M(gh_i^2|x)/m(gh_i^2|x) < L_1
$$

for all x, all i and $s > 5$. Then

$$
\begin{aligned}
A_i &\le 4L_1 \int_{R^p} M(g\|\nabla h_i\|^2|x) dx \\
&= 4L_1 \int_{R^p} \{1/C_f\} F(\|x-\theta\|) dx \int_{R^p} g(\theta) \|\nabla h_i(\theta)\|^2 d\theta \\
&= 4L_1 \int_{R^p} g(\theta) \|\nabla h_i(\theta)\|^2 d\theta \\
&= 8L_1 \frac{\pi^{p/2}}{\Gamma(p/2)} \int_0^\infty t^{p-1} G(t) \left\{ \frac{d}{dt} H_i(t) \right\}^2 dt,
\end{aligned}
$$

which goes to 0 as $i \to \infty$ by Theorem 2.3.

Next we consider B_i. $M(\nabla g|x)$ and $M(\nabla g h_i^2|x)$ at $x = 0$ are zero vectors because g and h_i^2 are function of $\|\theta\|$. So the integrand of B_i is bounded around $x = 0$. When we consider the asymptotic property of the integrand of B_i, note that there exists an L_2 such that $\eta|G'(\eta)/G(\eta)| \le L_2$ for all $\eta > 0$ under the Assumption **G1** because the regularly varying G with index α satisfies $\lim_{\eta\to\infty} \eta G'(\eta)/G(\eta) = \alpha$. Then we have

$$
\begin{aligned}
&\left|\frac{M(\nabla_j g|x)}{m(g|x)} - \frac{M(\nabla_j g h_i^2|x)}{m(gh_i^2|x)}\right| \\
&= \left|\int_{R^p} \frac{\theta_j}{\|\theta\|} G'(\|\theta\|) \left(\frac{1}{m(g|x)} - \frac{h_i^2}{m(gh_i^2|x)}\right) F(\|x-\theta\|)d\theta\right| \\
&\le \int_{R^p} |G'(\|\theta\|)| \left|\frac{1}{m(g|x)} - \frac{h_i^2}{m(gh_i^2|x)}\right| F(\|x-\theta\|)d\theta \\
&\le L_2 \int_{R^p} \frac{G(\|\theta\|)}{\|\theta\|} \left(\frac{1}{\sqrt{m(g|x)}} + \frac{h_i}{\sqrt{m(gh_i^2|x)}}\right) \left|\frac{1}{\sqrt{m(g|x)}} - \frac{h_i}{\sqrt{m(gh_i^2|x)}}\right| F(\|x-\theta\|)d\theta \\
&\le \frac{2L_2}{\sqrt{m(g|x)m(gh_i^2|x)}} \int_{R^p} \frac{G(\|\theta\|)}{\|\theta\|} \left|1 - \sqrt{\frac{m(g|x)}{m(gh_i^2|x)}} h_i\right| F(\|x-\theta\|)d\theta
\end{aligned}
$$

By the Cauchy-Schwartz inequality, Theorem A.2 and Corollary A.1 in Appendix, the right-hand side of the inequality above is less than

$$
\begin{aligned}
&2L_2 \left[\frac{m(g\|\theta\|^{-1}|x)}{m(g|x)m(gh_i^2|x)} \int_{R^p} \frac{G(\|\theta\|)}{\|\theta\|} \left(1 - \sqrt{\frac{M(g|x)}{m(gh_i^2|x)}} h_i\right)^2 F(\|x-\theta\|)d\theta\right]^{1/2} \\
&< \frac{L_3}{\sqrt{m(gh_i^2|x)}} \sqrt{G(\|x\|)} H_1(\|x\|)
\end{aligned}
$$

for sufficiently large $\|x\|$, some constant L_3 and $s > 5$. Hence there exists L_4 and L_5 such that the integrand of B_i is less than

$$
\min\{L_4, L_5 G(\|x\|) H_1^2(\|x\|)\}.
$$

By (24) in Theorem 2.3 and the dominated convergence theorem, B_i converges to 0 as $i \to \infty$.

□

A Appendix

A.1 The Blyth method

There are several versions of the Blyth method. For our purpose, a following version from Brown (1971) and Brown and Hwang (1982) is useful.

Theorem A.1. *Assume that there is an increasing sequence of proper densities such that* $\int_{\|\theta\|\geq 1} g_1(\theta)d\theta > c$ *for some positive* c *and* $\Delta_i \to 0$ *as* $i \to \infty$*. Then* δ_g *is admissible.*

Proof. Suppose that δ_g is inadmissible and let $R(\theta, \delta') \leq R(\theta, \delta_g)$ for all θ with strict inequality for some θ. Let $\delta'' = (\delta_g + \delta')/2$. Then, using Jensen's inequality,

$$\begin{aligned} R(\theta, \delta'') &= \int \|\delta''(x) - \theta\|^2 f(\|x - \theta\|)dx \\ &< \left(\int \|\delta_g(x) - \theta\|^2 f(\|x - \theta\|)dx + \int \|\delta'(x) - \theta\|^2 f(\|x - \theta\|)dx \right) \\ &= \left[R(\theta, \delta') + R(\theta, \delta_g) \right]/2 \leq R(\theta, \delta_g), \end{aligned}$$

for any θ. $R(\theta, \delta'')$ and $R(\theta, \delta_g)$ are both continuous functions of θ. Hence there exists an $\epsilon > 0$ such that $R(\theta, \delta'') < R(\theta, \delta_g) - \epsilon$ for $\|\theta\| \leq 1$. Then

$$\begin{aligned} \Delta_i &\geq \int_{R^p} [R(\theta, \delta_g) - R(\theta, \delta'')] g_i(\theta)d\theta \\ &\geq \int_{\|\theta\|\leq 1} [R(\theta, \delta_g) - R(\theta, \delta'')] g_1(\theta)d\theta \\ &\geq \epsilon c > 0, \end{aligned}$$

which contradicts $\Delta_i \to 0$. $\square$

A.2 The asymptotic behaviors of expected values

We give some results on the asymptotic behaviors of expected values when the location parameter diverges to infinity. Actually, in Section 3, we need an evaluation of the asymptotic behavior of expectation

$$E_x[\rho(\theta)] = \int_{R^p} \rho(\theta) f(\|\theta - x\|)d\theta$$

for sufficiently large $\|x\|$, where a random vector θ has the density function $f(\|\theta - x\|)$. This is the expected value with respect to the posterior distribution. Interchanging the roles of x and θ, in this appendix, we consider the asymptotic behavior of expectation

$$E_\theta[\rho(X)] = \int_{R^p} \rho(x) f(\|x - \theta\|)dx$$

for sufficiently large $\|\theta\|$, where a random vector X has the density function $f(\|x - \theta\|)$.

Now we make the following regularity conditions on the density f and the function ρ.

F1 There exist $r_0 > 0$, $L > 0$, and $s > 1$, such that $r^{p+s} f(r) \le L$ for all $r \ge r_0$.

B1 $\rho(x)$ is written as $\rho(x) = \varrho(\|x\|)$, where $\varrho(r)$ is continuously differentiable in $r > 0$.

B2 There exists $r_1 \ge 1$ and $t_1 \le t_2$ such that $\varrho(r) > 0$ and $t_1 \le r\varrho'(r)/\varrho(r) \le t_2$ for all $r \ge r_1$.

The following lemma is useful. The proof, based on the integration of $(\log \varrho(r))' = \varrho'(r)/\varrho(r)$, is easy and omitted.

Lemma A.1. *Under the assumption **B2***

$$(z/y)^{t_1} \le \varrho(z)/\varrho(y) \le (z/y)^{t_2}$$

for any $z > y \ge r_1$. Moreover

$$\limsup_{y\to\infty} \sup_{\alpha y \le z \le \beta y} \varrho(z)/\varrho(y) \le \max(\alpha^{t_1}, \beta^{t_2})$$

for any $0 < \alpha < 1 < \beta$.

We now state the following theorem concerning the asymptotic behavior of $E_\theta[\rho(X)]$ for large $\|\theta\|$.

Theorem A.2. *Assume* **F1**, **B1** *and* **B2**. *If $s > \max(1, -t_1 - p, t_2)$ and $\int_0^1 r^{p-1}|\varrho(r)|dr < \infty$, then there exists $\epsilon > 0$ (say $\epsilon = \min(1, s + t_1 + p)/4$) such that*

$$\|\theta\|^\epsilon \left| E_\theta[\rho(X)] - \rho(\theta) \right| < C\rho(\theta) \tag{26}$$

for $\|\theta\| \ge 2\max(r_0, r_1)$. Moreover C depends on ρ (or ϱ) only through r_1, t_1, t_2 and $\{\varrho(r_1)\}^{-1} \int_0^{r_1} r^{p-1}|\varrho(r)|dr$.

Proof. Fix $0 < \nu < 1$ (set $\nu = 1/2$ finally). Define

$$\begin{aligned} V_\nu &= \{x : \|x - \theta\| \le \nu\|\theta\|\} \\ V'_\nu &= \{x : (1-\nu)\|\theta\| \le \|x\| \le (1+\nu)\|\theta\|\}. \end{aligned}$$

Clearly $V_\nu \subset V'_\nu$. Then

$$\begin{aligned} &\|\theta\|^\epsilon \left| E[\rho(X) - \rho(\theta)] \right| \\ &\le \|\theta\|^\epsilon \left(\int_{V_\nu} + \int_{V_\nu^C} \right) |\rho(x) - \rho(\theta)|\, f(\|x - \theta\|)dx \end{aligned}$$

$$\begin{aligned}
&\le \|\theta\|^{\epsilon} \int_{V_\nu} |\rho(x) - \rho(\theta)|\, f(\|x-\theta\|)dx \\
&\quad + \|\theta\|^{\epsilon}\rho(\theta) \int_{V_\nu^C} f(\|x-\theta\|)dx + \|\theta\|^{\epsilon} \int_{V_\nu^C} |\rho(x)| f(\|x-\theta\|)dx \\
&= I_1 + I_2 + I_3 \quad \text{(say)}.
\end{aligned} \tag{27}$$

Consider the first integral I_1. If $s > 1$, then $m_1 = \int_{R^p} \|x-\theta\| f(\|x-\theta\|)dx$ is finite. Therefore for $\|\theta\| \ge (1-\nu)^{-1} r_1$ we have

$$\begin{aligned}
&\|\theta\|^{\epsilon} \int_{V_\nu} |\rho(x) - \rho(\theta)|\, f(\|x-\theta\|)dx \\
&\quad = \|\theta\|^{\epsilon} \int_{V_\nu} |(x-\theta)' \nabla\rho(x^*)| f(\|x-\theta\|)dx,\ x^* \in V_\nu \\
&\quad \le m_1 \|\theta\|^{\epsilon} \sup_{x \in V_\nu} \|\nabla \rho(x)\| \\
&\quad \le m_1 \|\theta\|^{\epsilon} \sup_{x \in V_\nu} |\varrho'(\|x\|)| \\
&\quad \le m_1 \|\theta\|^{\epsilon-1} \sup_{x \in V_\nu'} \frac{\|\theta\|}{\|x\|} \sup_{x \in V_\nu'} \frac{\varrho(\|x\|)}{\varrho(\|\theta\|)} \sup_{x \in V_\nu'} \frac{\|x\| |\varrho'(\|x\|)|}{\varrho(\|x\|)} \times \rho(\theta) \\
&\quad \le \frac{m_1}{1-\nu} \max\left((1-\nu)^{t_1}, \{1+\nu\}^{t_2}\right) \max(|t_1|, |t_2|) \times \rho(\theta)
\end{aligned}$$

for $0 < \epsilon < 1$. Therefore we have $I_1 \le C_1 \rho(\theta)$ for some C_1.

Now we consider the integral outside of V_ν. We only consider $\|\theta\| \ge \max(\nu^{-1} r_0, r_1)$. Then for $x \in V_\nu^C$

$$\|x - \theta\| \ge \nu \|\theta\| \ge r_0.$$

Therefore we have, for $0 \le \alpha < s$

$$\begin{aligned}
\int_{V_\nu^C} \|x\|^{\alpha} f(\|x-\theta\|)dx &\le \int_{V_\nu^C} \{\|x-\theta\| + \|\theta\|\}^{\alpha} f(\|x-\theta\|)dx \\
&\le (1+1/\nu)^{\alpha} \int_{V_\nu^C} \|x-\theta\|^{\alpha} f(\|x-\theta\|)dx \\
&\le (1+1/\nu)^{\alpha} c_p L \int_{\nu\|\theta\|}^{\infty} r^{-s+\alpha-1} dr \\
&= (1+1/\nu)^{\alpha} c_p L \frac{(\nu\|\theta\|)^{-s+\alpha}}{s-\alpha} \\
&\le C_2(\alpha) \|\theta\|^{\alpha-s},
\end{aligned} \tag{28}$$

where $C_2(\alpha) = (1+1/\nu)^{\alpha} c_p L \nu^{\alpha-s} (s-\alpha)^{-1}$. Hence for the second term I_2, if $s > 1$ and $0 < \epsilon < 1$, then $I_2 \le C_2(0)\rho(\theta)$.

We have seen that I_1 and I_2 are bounded from above assuming only $s > 1$.

The third term I_3 of (27) is more problematic. Write

$$\begin{aligned}
I_3 &= \|\theta\|^\epsilon \int_{V_\nu^C} |\rho(x)| f(\|x-\theta\|)dx \\
&\leq \|\theta\|^\epsilon \Bigg(\int_{V_\nu^C \cap \{\|x\|<r_1\}} + \int_{V_\nu^C \cap \{r_1 \leq \|x\| \leq \|\theta\|\}} \\
&\qquad\qquad + \int_{V_\nu^C \cap \{\|x\|>\|\theta\|\}} \Bigg) |\rho(x)| f(\|x-\theta\|)dx \\
&= I_{31} + I_{32} + I_{33} \qquad \text{(say.)}
\end{aligned}$$

We take care of I_{33} first. Since $\varrho(r)r^{-t_2}$ is monotone nonincreasing for $r \geq r_1$, $\rho(x)\|x\|^{-t_2} \leq \rho(\theta)\|\theta\|^{-t_2}$ for $\|x\| > \|\theta\| (\geq r_1)$. Therefore we have,

$$I_{33} \leq \|\theta\|^{\epsilon - t_2} \rho(\theta) \int_{V_\nu^C \cap \{\|x\|>\|\theta\|\}} \|x\|^{t_2} f(\|x-\theta\|)dx.$$

If $0 \leq t_2 < s$, as in (28)

$$\begin{aligned}
\int_{V_\nu^C \cap \{\|x\|>\|\theta\|\}} \|x\|^{t_2} f(\|x-\theta\|)dx &\leq \int_{V_\nu^C} \|x\|^{t_2} f(\|x-\theta\|)dx \\
&\leq C_2(t_2)\|\theta\|^{t_2 - s}
\end{aligned}$$

and if $t_2 < 0$,

$$\begin{aligned}
\int_{V_\nu^C \cap \{\|x\|>\|\theta\|\}} \|x\|^{t_2} f(\|x-\theta\|)dx &\leq \|\theta\|^{t_2} \int_{V_\nu^C} f(\|x-\theta\|)dx \\
&\leq C_2(0)\|\theta\|^{t_2 - s}.
\end{aligned}$$

Hence $I_{33} \leq C_{33}\rho(\theta)$ where $C_{33} = \max(C_2(t_2), C_2(0))$.

Next we consider I_{31}. For $\|\theta\| \geq \max(\nu^{-1}r_0, r_1)$ and $x \in V_\nu^C$

$$f(\|x-\theta\|) \leq L\|x-\theta\|^{-p-s} \leq L(\nu\|\theta\|)^{-p-s}. \tag{29}$$

Therefore

$$I_{31} \leq \|\theta\|^\epsilon L \nu^{-p-s} \|\theta\|^{-p-s} \int_{\|x\| \leq r_1} \rho(x)dx.$$

Note that by simple change of variables we have

$$\frac{\partial}{\partial r} \int_{\|x\| \leq r} dx = c_p r^{p-1}.$$

Then

$$\int_{\|x\|\leq r_1} |\rho(x)|dx = c_p \int_0^{r_1} r^{p-1}|\varrho(r)|dr.$$

Therefore

$$I_{31} \leq C_* \|\theta\|^{\epsilon-p-s} \int_0^{r_1} r^{p-1}|\varrho(r)|dr,$$

where $C_* = L\nu^{-p-s}c_p$. On the other hand for $\|\theta\| \geq r_1$, $\rho(\theta) = \varrho(\|\theta\|)$ is bounded from below as

$$\varrho(r_1)r_1^{-t_1}\|\theta\|^{t_1} \leq \varrho(\|\theta\|).$$

Therefore

$$I_{31} \leq \|\theta\|^{\epsilon-p-s-t_1} \times C_* \frac{r_1^{t_1}}{\varrho(r_1)} \int_0^{r_1} r^{p-1}|\varrho(r)|dr \times \rho(\theta).$$

Hence if $s > -t_1 - p$, then we can choose $\epsilon > 0$ (say $\epsilon = (p + s + t_1)/4$) such that $\epsilon - p - s - t_1 < 0$ and hence $I_{31} \leq C_{31}\rho(\theta)$ where

$$C_{31} = C_* \frac{r_1^{t_1}}{\varrho(r_1)} \int_0^{r_1} r^{p-1}|\varrho(r)|dr.$$

Finally we consider I_{32}. Note $\varrho(r) \leq \varrho(\|\theta\|)\|\theta\|^{-t_1}r^{t_1}$ for $r_1 \leq r \leq \|\theta\|$ and (29). Then

$$\begin{aligned} I_{32} &\leq \|\theta\|^{\epsilon-t_1} L\nu^{-p-s}\|\theta\|^{-p-s}\varrho(\|\theta\|) \int_{r_1\leq\|x\|\leq\|\theta\|} \|x\|^{t_1}dx \\ &\leq \|\theta\|^{\epsilon-p-s-t_1} \times C_* \int_{r_1}^{\|\theta\|} r^{p+t_1-1}dr \times \rho(\theta). \end{aligned}$$

Consider the integral $Q = \int_{r_1}^{\|\theta\|} r^{p+t_1-1}dr$. If $p + t_1 < 0$, then

$$Q \leq r_1^{t_1+p}/(-t_1 - p).$$

Therefore as in the case of I_{31}, if $s > -t_1 - p$, then we can choose $\epsilon > 0$ (say $\epsilon = (p + s + t_1)/4$) such that $\epsilon - p - s - t_1 < 0$ and hence

$$I_{32} \leq r_1^{\epsilon-s} C_{32}\rho(\theta) \leq C_{32}\rho(\theta)$$

where $C_{32} = \{-1/(p + t_1)\}C_*$. If $p + t_1 \geq 0$,

$$Q = \int_{r_1}^{\|\theta\|} r^{p+t_1+\epsilon_3-1-\epsilon_3}dr \leq \frac{\|\theta\|^{p+t_1+\epsilon_3}r_1^{-\epsilon_3}}{p + t_1 + \epsilon_3}$$

for any $\epsilon_3 > 0$. Hence

$$I_{32} \le \frac{C_*}{p + t_1 + \epsilon_3} \|\theta\|^{\epsilon+\epsilon_3-s} \rho(\theta).$$

If $s > 1$, we can choose ϵ and ϵ_3 (say $\epsilon = \epsilon_3 = 1/4$) such that $\epsilon + \epsilon_3 - s < 0$ and hence

$$I_{32} \le C_{32} \rho(\theta),$$

where $C_{32} = (p + t_1 + \epsilon_3)^{-1} C_*$.

We have now confirmed that if $s > \max(1, -t_1 - p, t_2)$, there exist $\epsilon > 0$ and $C = C_1 + C_2 + C_{31} + C_{32} + C_{33}$, such that (26) folds for $\|\theta\| \ge \max(\nu^{-1} r_0, r_1, (1-\nu)^{-1} r_1)$ which equals to $2 \max(r_0, r_1)$ for $\nu = 1/2$. $\square$

In Section 3, we also need asymptotic behavior of the expectation of $\rho(X) \times h_i^\gamma(X)$ where $h_i(\theta) = H_i(\|\theta\|)$ given by (17) and $\gamma > 0$.

Corollary A.1. *Assume* **F1**, **B1** *and* **B2**. *If* $s > \max(1, \gamma - t_1 - p, t_2)$ *and* $\int_0^1 r^{p-1} |\varrho(r)| dr < \infty$, *there exists* $\epsilon > 0$ *(say* $\epsilon = \min(1, s + t_1 + p - \gamma)/4$*) such that*

$$\|\theta\|^\epsilon \left| E_\theta[\rho(X) h_i^\gamma(X)] - \rho(\theta) h_i^\gamma(\theta) \right| < C \rho(\theta) h_i^\gamma(\theta) \tag{30}$$

for $\|\theta\| \ge 2 d_1 \max(r_0, r_1, \eta_0)$. *Moreover* C *does not depend on* i.

Proof. Since we have

$$\eta \{\varrho(\eta) H_i^\gamma(\eta)\}' / \{\varrho(\eta) H_i^\gamma(\eta)\} = \eta \varrho'(\eta)/\varrho(\eta) + \gamma \eta H_i'(\eta)/H_i(\eta),$$

under Assumption **B2** and by part V of Theorem 2.1, for $\epsilon_1 = (s + t_1 + p - \gamma)/16 (> 0)$ there exists η_0 such that

$$t_1 - \gamma - \gamma \epsilon_1 \le \eta \{\varrho(\eta) H_i^\gamma(\eta)\}' / \{\varrho(\eta) H_i^\gamma(\eta)\} \le t_2$$

for all $\eta \ge \max(\eta_0, r_1)$. Then for $\epsilon = \min(1, s + t_1 + p - \gamma)/4 (> 0)$, (30) follows from Theorem A.2.

For $\lambda_1 = \max(\eta_0, r_1)$,

$$\frac{1}{H_i^\gamma(\lambda_1) \varrho(\lambda_1)} \int_0^{\lambda_1} r^{p-1} |H_i^\gamma(r) \varrho(r)| dr \le \frac{1}{H_1^\gamma(\lambda_1) \varrho(\lambda_1)} \int_0^{\lambda_1} r^{p-1} |\varrho(r)| dr$$

which implies that C does not depend on i. $\square$

Acknowledgment

This research is supported by the bilateral scientist exchanges program between Japan and Canada provided by JSPS in Japan and NSERC in Canada. I am indebted to Professor Zidek for his encouragement and hospitality during my sabbatical leave.

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