An admissibility proof using an adaptive sequence of
smoother proper priors approaching the target
Improper prior

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Abstract

We give a sufficient condition for the admissibility of generalized Bayes estimators of the location vector of spherically symmetric distribution under squared error loss. Compared to the known results for the multivariate normal case, our sufficient condition is very tight and is close to being necessary. In particular we establish the admissibility of generalized Bayes estimators with respect to the harmonic prior and priors with slightly heavier tails than the harmonic prior. The key to our proof is an adaptive sequence of smooth proper priors approaching an improper prior fast enough to establish admissibility.

1 Introduction

Let $X = (X_1, \ldots, X_p)'$ have a spherically symmetric density function $f(||x - \theta||)$ and consider estimation of a $p$-dimensional location parameter $\theta$ with a quadratic loss function $L(\theta, d) = (d - \theta)'(d - \theta) = ||d - \theta||^2$. Therefore an estimator $\delta(X)$ is evaluated using the risk function

$$R(\theta, \delta) = E_\theta \left[ ||\delta(X) - \theta||^2 \right] = \int_{R^p} ||\delta(x) - \theta||^2 f(||x - \theta||)dx.$$  

An estimator $\delta$ is said to be admissible if no estimator $\delta'$ exists such that $R(\theta, \delta') \leq R(\theta, \delta)$ for all $\theta$ with strict inequality for some $\theta$. Hence admissibility is a desirable property for estimators. It is well-known that any proper Bayes estimator is admissible under very mild conditions. In many cases, however, a target estimator is generalized Bayes (gBayes), with respect to an improper prior like the Lebesgue measure. There is no guarantee that any gBayes estimator is admissible.

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A famous sufficient condition for admissibility of gBayes estimator has been given by Blyth (1951). A version of the Blyth result is the following. Let \( g(\mu) \) be the target improper prior density and \( g_1 \cdot g_2 \cdot \cdots \cdot g \), an increasing sequence of proper prior densities approaching \( g \). Each \( g_i \) is not necessarily normalized, so just satisfies \( \int_{\Theta} g_i(\theta) d\theta < \infty \) for any fixed \( i \). Let \( \delta_g \) and \( \delta_{g_i} \) be the gBayes estimator with respect to \( g(\mu) \) and the proper Bayes estimator with respect to \( g_i(\mu) \), respectively. The non-standardized Bayes risk difference between \( \delta_g \) and \( \delta_{g_i} \) with respect to \( g_i(\mu) \) is given by

\[
\Delta_i = \int_{\Theta} [R(\theta, \delta_g) - R(\theta, \delta_{g_i})] g_i(\theta) d\theta.
\]

Blyth (1951) showed that if \( \Delta_i \to 0 \) as \( i \to \infty \), \( \delta_g \) is admissible. (See Theorem A.1 in Appendix.) Therefore a good choice of the sequence of proper priors approaching the target prior is the key to finding admissible gBayes estimators. As Berger (1985) pointed out, however, “Indeed, in general, very elaborate (and difficult to work with) choices of the \( g_i \) are needed.” For example, when \( p = 1 \) under normality and spherical symmetry, Blyth (1951) and Stein (1959) showed that the most natural estimator \( X \), which is gBayes with respect to \( g(\mu) = 1 \), is admissible by using a sequence of conjugate priors \( g_i(\theta) = \exp(-\theta^2/i) \) and by using \( g_i(\theta) = (1 + \theta^2/i)^{-1} \), respectively. These are relatively comprehensible choices. But when \( p = 2 \), neither a sequence \( g_i(\theta) = \exp(-||\theta||^2/i) \) nor a sequence \( g_i(\theta) = (1 + \theta^2/i)^{-1} \) works to show the admissibility of \( X \) under normality. Under spherically symmetry, James and Stein (1961) showed for \( p = 2 \) that \( g_i(\theta) = h_i^2(\theta) \) works where

\[
h_i(\theta) = \begin{cases} 
1 & ||\theta|| \leq 1 \\
1 - \frac{\log ||\theta||}{\alpha(i, ||\theta||)} & 1 \leq ||\theta|| \leq i/2 \\
\alpha(i, ||\theta||) & ||\theta|| > i/2 
\end{cases}
\]

and \( \alpha(i, ||\theta||) \) is chosen so that, for fixed \( \theta \), \( \alpha(i, ||\theta||)||\theta||^{-1}\{\log ||\theta||\}^{-1} \to 1 \) as \( i \to \infty \) and \( h_1 \leq h_2 \leq \cdots \leq 1 \). On the other hand, \( X \) for \( p \geq 3 \), is inadmissible as shown by Stein (1956) under normality and Brown (1966) in quite a general setting. Therefore, in general, we would like to know the mechanisms for discrimination between admissibility and inadmissibility for gBayes estimators. In this paper, we will investigate a sufficient condition for the admissibility of gBayes estimators with respect to spherically symmetric priors.

Under normality, Brown (1971) gave a powerful condition for admissibility as follows. Let \( G(||\theta||) \) be a spherically symmetric target prior density. Then the marginal density
is also spherically symmetric as \( m(||x||) = \int f(||x - \theta||)G(||\theta||)d\theta \). Brown (1971) showed that if the gBayes estimator with respect to \( G(||\theta||) \) has a finite risk and

\[
\int_1^\infty r^{1-p} m(r)^{-1} dr = \infty
\]

then it is admissible. Since \( m(r) \sim G(r) \) for large \( r \) under suitable, mild conditions as shown in Maruyama and Takemura (2006), the sufficient condition above reduces to

\[
\int_1^\infty r^{1-p} G(r)^{-1} dr = \infty.
\]

Brown also showed that if the integral in (2) is finite, the gBayes estimator is inadmissible. Needless to say, Brown (1971) dealt with quite general priors (which are permitted to have a non-differentiable density, to have some holes on \( \mathbb{R}^p \) and not to be spherically symmetric) and gave a general sufficient condition for them. Unfortunately even if we assume that the target prior has a differentiable density and that the support is \( \mathbb{R}^p \) in Brown’s (1971) condition, we do not find an easier proof than Brown (1971) and his choice of the sequence is still extraordinarily complicated.

On the other hand, Brown and Hwang (1982) consider estimation of the natural mean vector of an exponential family under a quadratic loss function and so the intersection of their setting and ours is the normal case. Brown and Hwang (1982) give a sufficient condition for gBayes estimators to be admissible when the target prior density \( g(\theta) = G(||\theta||) \) is differentiable. Their ingenuity lies in the decomposition of \( \Delta_i \) given by (1), a result using the triangle and Cauchy-Schwartz inequalities, i.e.

\[
\Delta_i \leq 8 \int_{\mathbb{R}^p} g(\theta) ||\nabla h_i(\theta)||^2 d\theta + 2 \int_{\mathbb{R}^p} \left\| \frac{m(\nabla g|x)}{m(g|x)} - \frac{m(\nabla g h_i^2|x)}{m(g h_i^2|x)} \right\|^2 m(g h_i^2|x) dx
\]

\[
= A_i + B_i
\]

where \( g_i = g h_i^2 \), \( m(\psi|x) = \int_{\mathbb{R}^p} \psi(\theta)f(||\theta - x||)d\theta \) and the gradient of a function \( \rho(x) \) is denoted by

\[
\nabla \rho(x) = \left( \frac{\partial}{\partial x_1} \rho(x), \ldots, \frac{\partial}{\partial x_p} \rho(x) \right)'.
\]

Hence the problem reduces to the simultaneous minimization problem of \( A_i \) and \( B_i \). Brown and Hwang (1982) showed that when the sequence is chosen as

\[
h_i(\theta) = \begin{cases} 
1 & ||\theta|| \leq 1 \\
1 - \log ||\theta||/\log i & 1 \leq ||\theta|| \leq i \\
0 & ||\theta|| > i,
\end{cases}
\]

(3)
$A_i$ and $B_i$ go to 0 as $i \to \infty$ if
\[
\int_1^{\infty} \frac{r^{p-3}G(r)dr}{(\log(r + 2))^2} < \infty
\]
and
\[
\int_{R^p} m \left( g \left\| \nabla g \right\| \frac{m(\nabla g)}{m(g)} \right)^2 dx < \infty,
\]
respectively. Needless to say, when $A_i$ and $B_i$ go to 0 as $i \to \infty$, the gBayes estimator is admissible by the Blyth method. They called (4) and (5) the “growth condition” and “asymptotic flatness condition”, respectively. We see that their method of proof is much more transparent than Brown’s (1971) and their sequence in (3) is simpler.

However the growth condition may be weaker than Brown’s (1971) condition given by (2) e.g. $G(\|\theta\|) = \|\theta\|^{2-p} \log(\|\theta\| + 2)$, which is slightly heavier than $\|\theta\|^{2-p}$, satisfies (2), but not the growth condition. The reason seems to be that the sequence (3) does not depend on the target prior density $g$ but is optimized for $g(\theta) \leq \|\theta\|^{2-p}$ for sufficiently large $\|\theta\|$. Furthermore if $G(\|\theta\|) \to \infty$ around the origin like $\|\theta\|^{2-p}$, it does not satisfy the asymptotic flatness condition. The reason found in their method of bounding $B_i$ in order to apply the dominated convergence theorem, is very rough around the origin. Moreover the lack of power in Brown and Hwang (1982) stems from their choice of the sequence; $h_i$ given in (3) is non-differentiable at $\|\theta\| = 1$ and truncated at $\|\theta\| = i$. When we deal with $A_i$, the truncated sequence does not cause trouble. But the truncated sequence generally does cause trouble when dealing with $B_i$.

In this paper, I naturally extend Brown-Hwang’s decomposition method to the spherically symmetric case and give as strong a condition as that of Brown (1971) under normality. In Section 2, I consider a minimization problem, and show for the corresponding term $A_i$ in Brown and Hwang (1982),
\[
\inf_{h} \int_0^{\infty} \{h'(\eta)\}^2 \eta^{p-1} G(\eta)d\eta = 0
\]
under some constraints. As an alternative to (3), I propose a smoother sequence for a solution of the problem
\[
H_i(\eta) = \frac{\int_\eta^{\infty} e^{(\eta-r)/i} \beta(r)dr}{\int_\eta^{\infty} \beta(r)dr} \quad (i = 1, 2, \ldots)
\]
where

\[
\beta(r) = -\frac{d}{dr} \left\{ \left( \int_1^{2+r} \frac{s^{1-p}}{G(s)} ds \right)^{-1} \right\} = \frac{(r + 2)^{1-p}/G(2 + r)}{(\int_1^{2+r} \{s^{1-p}/G(s)\} ds)^2},
\]

which works very well when \( \int_1^{\infty} \{s^{1-p}/G(s)\} ds = \infty \). This choice of the adaptive sequence to \( G \) is stimulated by the sequence in Zidek (1970), which was however truncated and non-differentiable. In Section 3, we show that our \( H_i \) also works well for proving that the corresponding \( B_i \) approaches 0 as \( i \to \infty \) in spherically symmetric case. As a result, we can prove a strong sufficient condition for the admissibility of gBayes estimators by using an adaptive sequence of proper priors \( G(||\theta||)H^2_1(||\theta||) \) which approaches the target improper prior \( G(||\theta||) \). In particular, we show that the gBayes estimators with respect to the harmonic prior \( G(||\theta||) = ||\theta||^{2-p} \) and with respect to a prior with a slightly heavier tail

\[
G(||\theta||) = ||\theta||^{2-p} \log(||\theta|| + c), \quad c > 1,
\]

are admissible under mild regularity conditions on \( f \).

Brown (1979) considered a more general problem than ours: estimation of \( \theta \) for a general density \( p(x - \theta) \) and a general loss function \( W(\delta - \theta) \). He conjectured that the prior \( g(\theta) \sim ||\theta||^a \) with \( a \leq 2 - p \) for sufficiently large \( ||\theta|| \) leads to admissibility, regardless of the density \( p \) and the loss \( W \). However there has been no exact results about admissibility in this type of setting unless normality and quadratic loss function are assumed. Hence our results support Brown’s (1979) conjecture for the case of spherically symmetric family and a quadratic loss function.

The companion paper of Maruyama and Takemura (2006) deals with the same problem and gives a sufficient condition for admissibility without the assumption that the target prior is regularly varying. However the results in Maruyama and Takemura (2006) do not necessarily include the ones in this paper. Adaptive sequence of proper priors of the type suggested by Zidek (1970) as well as the assumption of the regularly varying prior yield more elegant results than in Maruyama and Takemura (2006).

2 A minimization problem

In this section, when a nonnegative function \( k(\eta) \) satisfies

\[
\int_0^1 k(\eta)d\eta < \infty
\]

...
and
\[ \int_1^\infty k(\eta)d\eta = \infty, \] (10)
we consider a minimization problem
\[ \inf_h \int_0^\infty \{h'(\eta)\}^2 k(\eta)d\eta = 0 \] (11)
subject to
\[ \int_0^\infty h^2(\eta)k(\eta)d\eta < \infty. \] (12)
In Section 3, we set \( k(\eta) = \eta^{p-1}G(\eta) \) where \( G(\|\theta\|) \) is our target prior density. This type of minimization problem has been famous in mathematical physics. See Rukhin (1995) for the details. A very well-known sufficient condition on \( k(\eta) \) to satisfy (11) is
\[ \int_1^\infty \frac{d\eta}{k(\eta)} = \infty. \] (13)
Indeed when (13) is satisfied, we define \( h_i (i = 1, \ldots) \) as
\[ h_i(\eta) = \begin{cases} 1 & 0 < \eta < 1/2 \\ \frac{\int_s^1 \{1/k(s)\}ds}{\int_1^{1/2} \{1/k(s)\}ds} & 1/2 \leq \eta < i \\ 0 & \eta \geq i, \end{cases} \] (14)
and easily find that
\[ \int_0^\infty \{h'_i(\eta)\}^2 k(\eta)d\eta = \frac{1}{\int_{1/2}^i \{1/k(s)\}ds}, \] (15)
which approaches 0 as \( i \to \infty \). Since \( h_i(\eta) \) is truncated at \( \eta = i \), \( \int_0^\infty h_i^2(\eta)k(\eta)d\eta < \infty \) is guaranteed.

In the statistical context, this type of sequence has been considered by Stein (1965), Zidek (1970) and Brown (1971). However, we will have to apply the same sequence for (11) to another minimization problem (\( \inf B_i = 0 \) as explained in Section 1). It is very hard to deal with a truncated and non-differentiable \( h_i(\eta) \) like (14) in such a simultaneous minimization problem. Here we produce a differentiable and non-truncated sequence for our purpose.
We assume that \( k(\eta) \) is continuously differentiable and regularly varying with index \( \alpha \), that is,

\[
\lim_{\eta \to \infty} k(\eta x)/k(\eta) = x^\alpha
\]

(16)

for any \( x > 0 \). When \( k(\eta) \) satisfies (16), we sometimes use the notation \( k(\eta) \in \text{RV}_{\alpha} \). A typical \( k(\eta) \) is \( \eta^\alpha \{ \log(\eta + 2) \}^\beta \) for any \( \beta \). Under (10) and (13), we have only to consider the case \(-1 \leq \alpha \leq 1\).

We now define functions \( H_i(\eta), i = 1, 2, \ldots \) by

\[
H_i(\eta) = \frac{\int_0^\infty e^{(\eta-r)/i} \beta(r)dr}{\int_\eta^\infty \beta(r)dr}
\]

where

\[
\beta(r) = -\frac{d}{dr} \left\{ \left( \int_1^{2+r} \frac{1}{k(s)} ds \right)^{-1} \right\} = \frac{1/k(2+r)}{(\int_1^{2+r} \{1/k(s)\} ds)^2}.
\]

(17)

The properties of \( \beta \) and \( H_i \) are given in the following theorems.

**Theorem 2.1.** (I) \( \beta(r) \in \text{RV}_{\alpha-2}, \int_r^\infty \beta(s)ds \in \text{RV}_{\alpha-1} \) and \( \beta'(r) \in \text{RV}_{\alpha-3} \).

(II) \( \lim_{r \to \infty} r \beta(r)/\int_r^\infty \beta(s)ds = \alpha - 1 \) and \( \lim_{r \to \infty} r \beta'(r)/\beta(r) = \alpha - 2 \).

(III) \( \beta(r)/\int_r^\infty \beta(s)ds \) is bounded for \( r \geq 0 \).

**Proof.** See Proposition 1.7 of Geluk and de Haan (1987) for part I and II.

By part I, \( \beta'(r)/\int_r^\infty \beta(s)ds \in \text{RV}_{-1} \) and hence \( \beta(r)/\int_r^\infty \beta(s)ds \to 0 \) as \( r \to \infty \). Since \( \int_0^\infty \beta(r)dr < \infty \) by (18) and \( k(r) \) is continuous, \( \{\beta(r)/\int_r^\infty \beta(s)ds\}_{r=0} \) is bounded. \( \square \)

By part II, there exists \( r_0 \) such that \( \beta'(r) \leq 0 \) for all \( r \geq r_0 \). By redefining \( \beta(r) \) as \( \beta(r + r_0) \), we have \( \beta(r) \) which is nondecreasing for \( r > 0 \).

**Theorem 2.2.** (I) \( 0 \leq H_1(\eta) \leq H_2(\eta) \leq \cdots \leq 1 \). For any fixed \( \eta \), \( \lim_{i \to \infty} H_i(\eta) = 1 \).

(II) For any fixed \( i \), \( \lim_{\eta \to \infty} \int_\eta^\infty \beta(r)dr \beta(\eta)^{-1} H_i(\eta) = i \).

(III) For any fixed \( \eta \), \( \lim_{i \to \infty} H_i'(\eta) = 0 \).

(IV) \( |H_i'(\eta)| < 2 \beta(\eta)/\int_\eta^\infty \beta(r)dr \) for all \( \eta > 0 \).
(V) For any $\epsilon > 0$, there exists $\eta_0$ such that $-1 - \epsilon < \eta H_i'(\eta)/H_i(\eta) \leq 0$ for all $\eta \geq \eta_0$ and for all $i$.

Proof. It is obvious that $0 \leq H_i(\eta) \leq 1$ and $H_i(\eta)$ is increasing in $i$. For fixed $\eta$, $H_i(\eta) \uparrow 1$ by the monotone convergence theorem.

By integration by parts, the numerator of $H_i(\eta)$ is written as

$$\int_{\eta}^{\infty} e^{(\eta - r)/i} \beta(r) dr = i\beta(\eta) + i \int_{\eta}^{\infty} e^{(\eta - r)/i} \beta'(r) dr.$$

Therefore

$$H_i(\eta) = \frac{i \beta(\eta)}{\int_{\eta}^{\infty} \beta(r) dr} + \frac{i \int_{\eta}^{\infty} e^{(\eta - r)/i} \beta'(r) dr}{\int_{\eta}^{\infty} \beta(r) dr}.$$

(20) divided by (17) is

$$1 = \frac{i \beta(\eta)}{H_i(\eta) \int_{\eta}^{\infty} \beta(r) dr} + \frac{i \int_{\eta}^{\infty} e^{(\eta - r)/i} \beta'(r) dr}{\int_{\eta}^{\infty} \beta(r) dr}.$$

Since $\beta'(r)/\beta(r) \rightarrow 0$ by part II of Theorem 2.1, the second term of the above equation for fixed $i$, converges to 0 as $\eta \rightarrow \infty$ by the L’Hospital theorem.

Using (19) again, differentiation of the numerator of $H_i(\eta)$ gives

$$\left( \int_{\eta}^{\infty} e^{(\eta - r)/i} \beta(r) dr \right)' = \frac{1}{i} \int_{\eta}^{\infty} e^{(\eta - r)/i} \beta'(r) dr - \beta(\eta) = \int_{\eta}^{\infty} e^{(\eta - r)/i} \beta'(r) dr.$$

Therefore

$$H_i'(\eta) = \frac{\beta(\eta) \int_{\eta}^{\infty} e^{(\eta - r)/i} \beta(r) dr}{(\int_{\eta}^{\infty} \beta(r) dr)^2} - \frac{\int_{\eta}^{\infty} e^{(\eta - r)/i} \{-\beta'(r)\} dr}{\int_{\eta}^{\infty} \beta(r) dr}.$$

(21)

Note that $-\beta'(r) \geq 0$ by redefinition of $\beta$. Each term of the right hand side of (21) is nondecreasing in $i$ and hence by the monotone convergence theorem

$$\lim_{i \rightarrow \infty} H_i'(\eta) = \frac{\beta(\eta) \int_{\eta}^{\infty} \beta(r) dr}{(\int_{\eta}^{\infty} \beta(r) dr)^2} - \frac{\int_{\eta}^{\infty} \{-\beta'(r)\} dr}{\int_{\eta}^{\infty} \beta(r) dr} = 0.$$

Furthermore we have

$$|H_i'(\eta)| < \left| \frac{\beta(\eta) \int_{\eta}^{\infty} e^{(\eta - r)/i} \beta(r) dr}{(\int_{\eta}^{\infty} \beta(r) dr)^2} \right| + \left| \frac{\int_{\eta}^{\infty} e^{(\eta - r)/i} \{-\beta'(r)\} dr}{\int_{\eta}^{\infty} \beta(r) dr} \right|$$
By II of Theorem 2.1 the right hand side converges to \(-1\). This implies that for any \(\epsilon > 0\) there exists \(\eta_0\) such that \(\eta H'_i(\eta)/H_i(\eta) > -1 - \epsilon\) for all \(\eta \geq \eta_0\) and for all \(i\). Finally we will prove that \(H'_i(\eta) \leq 0\) for sufficiently large \(\eta\) independent of \(i\). By II of Theorem 2.1,

\[
\frac{\eta H'_i(\eta)}{H_i(\eta)} = \eta \left( \frac{\beta(\eta)}{\int_\eta^\infty \beta(r)dr} + \frac{\int_\eta^\infty e^{-r/i} \beta'(r)dr}{\int_\eta^\infty e^{-r/i} \beta(r)dr} \right)
\]

and hence \(\beta(\eta)/\int_\eta^\infty \beta(r)dr\) is eventually nonincreasing. Hence by redefining \(\eta_0\) if necessary, we can assume that \(\beta(\eta)/\int_\eta^\infty \beta(r)dr\) is monotone nonincreasing for \(\eta \geq \eta_0\). By integration by parts on the numerator of the first term in (21), we have

\[
\int_\eta^\infty e^{-r/i} \beta(r)dr = e^{-\eta/i} \int_\eta^\infty \beta(r)dr - i^{-1} \int_\eta^\infty e^{-r/i} \left\{ \int_r^\infty \beta(s)ds \right\}dr
\]

and hence

\[
\left\{ \frac{i \int_\eta^\infty \beta(r)dr}{\int_\eta^\infty e^{(\eta-r)/i} \left\{ \int_r^\infty \beta(s)ds \right\}dr} \right\} H'_i(\eta)
\]

\[-\frac{\beta(\eta)}{\int_\eta^\infty \beta(r)dr} + \frac{\int_\eta^\infty e^{-r/i} \beta(r)dr}{\int_\eta^\infty e^{-r/i} \left\{ \int_r^\infty \beta(s)ds \right\}dr} \]

\[-\frac{\beta(\eta)}{\int_\eta^\infty \beta(r)dr} + \frac{\int_\eta^\infty \beta(r) / \int_r^\infty \beta(s)ds e^{-r/i} \left\{ \int_r^\infty \beta(s)ds \right\}dr}{\int_\eta^\infty e^{-r/i} \left\{ \int_r^\infty \beta(s)ds \right\}dr}
\]

\[-\frac{\beta(\eta)}{\int_\eta^\infty \beta(r)dr} + \sup_{t \geq \eta} \frac{\beta(t)}{\int_\eta^\infty \beta(r)dr},
\]

which is zero for \(\eta \geq \eta_0\). Hence we find that \(H'_i(\eta) \leq 0\) for all \(\eta \geq \eta_0\) and for all \(i\).
Now we show that $H_i(\eta)$ works very well for the minimization problem (11).

**Theorem 2.3.** Assume $\int_1^\infty \{1/k(\eta)\}d\eta = \infty$. Then $H_i(\eta)$ given by (17) and (18) satisfies

$$\lim_{i \to \infty} \int_0^\infty \left\{ \frac{d}{d\eta} H_i(\eta) \right\}^2 k(\eta)d\eta = 0$$  \hspace{1cm} (23)

and

$$\int_0^\infty H_i^2(\eta)k(\eta)d\eta < \infty$$  \hspace{1cm} (24)

for fixed $i$.

**Proof.** Note that

$$\int_a^\infty \frac{\beta(\eta)}{\int_\eta^\infty \beta(r)dr} \cdot k(\eta)d\eta = \int_a^\infty \frac{k(\eta)}{k(\eta) + 2} \frac{d}{dr} \left\{ - \left[ \int_1^{2+r} \frac{1}{k(s)} ds \right]^{-1} \right\} \bigg|_{r=\eta} d\eta$$

$$\leq \sup_{i \geq a} \frac{k(\eta)}{k(\eta) + 2} \left[ \int_1^{2+a} \frac{1}{k(s)} ds \right]^{-1}$$

for $a > 0$. Using part IV of Theorem 2.2 and part III of Theorem 2.1, we have

$$\int_0^\infty \left\{ \frac{d}{d\eta} H_i(\eta) \right\}^2 k(\eta)d\eta$$

$$\leq 4 \left( \int_0^1 + \int_1^{\infty} \right) \left\{ \frac{\beta(\eta)}{\int_\eta^\infty \beta(r)dr} \right\} \cdot k(\eta)d\eta$$

$$\leq 4 \sup_{0 \leq t \leq 1} \frac{\beta(\eta)}{\int_\eta^\infty \beta(r)dr} \int_0^1 k(\eta)d\eta + 4 \int_1^{\infty} \left\{ \frac{\beta(\eta)}{\int_\eta^\infty \beta(r)dr} \right\} \cdot k(\eta)d\eta$$

$$< \infty$$

which guarantees (23) by the dominated convergence theorem together with part III of Theorem 2.2.

For (24), since $H_i(\eta) \leq i\beta(\eta)/\int_\eta^\infty \beta(s)ds$ for any $\eta > 0$ from (20), we have

$$\left( \int_0^{\eta_0} + \int_{\eta_0}^{\infty} \right) H_i^2(\eta)k(\eta)d\eta$$
\[ \leq \int_0^{r_0} k(\eta) d\eta + i^2 \int_{r_0}^{\infty} \left\{ \frac{\beta(\eta)}{\int_{1}^{\infty} \beta(r) dr} \right\}^2 k(\eta) d\eta \]

< \infty,

for any fixed \( i \).

\[ \square \]

### 3 Admissibility

In this section, we give a sufficient condition for admissibility of the \( g \)-Bayes estimator with respect to a spherically symmetric target prior density \( g(\theta) = G(||\theta||) \). The assumptions on the behavior of \( G \) and \( f \) are following.

**F1** There exist \( r_0 > 0, L > 0, \) and \( s > 1 \), such that \( r^{p+s} f(r) \leq L \) for all \( r \geq r_0 \).

**G1** \( \eta G'(\eta)/G(\eta) \) is bounded for \( 0 < \eta < 1 \).

**G2** \( \int_0^{1} \eta^{p-1} G(\eta) d\eta < \infty \) and \( \int_0^{1} \eta^{p-1} |G'(\eta)| d\eta < \infty \).

**G3** \( \int_1^{\infty} \eta^{p-1} G(\eta) d\eta = \infty \).

**G4** \( G \) is continuous differentiable and regularly varying.

**FG1** \( \int_0^{\infty} r^{p-1} f(r) G(r) dr < \infty \) and \( \int_0^{\infty} r^{p-2} F(r) G(r) dr < \infty \).

From **G2** and **G3**, impropriety of \( G \) occurs at infinity. Notice that **G2** and **G3** correspond to the constraints (9) and (10) in Section 2.

The \( g \)-Bayes estimator \( \delta_g \) with respect to the improper density \( g(\theta) \) is written as

\[
\delta_g(x) = \frac{\int_{R^p} \theta f(||x - \theta||)g(\theta) d\theta}{\int_{R^p} f(||x - \theta||)g(\theta) d\theta}
\]

\[
= x + \frac{\int_{R^p} (\theta - x) f(||x - \theta||)g(\theta) d\theta}{\int_{R^p} f(||x - \theta||)g(\theta) d\theta}
\]

\[
= x + \frac{\int_{R^p} F(||x - \theta||) \nabla g(\theta) d\theta}{\int_{R^p} f(||x - \theta||)g(\theta) d\theta},
\]

which is well-defined if both \( \int_{R^p} F(||x - \theta||) \nabla g(\theta) d\theta \) and \( \int_{R^p} f(||x - \theta||)g(\theta) d\theta \) are integrable for all \( x \). These are guaranteed by the assumptions above.

Write

\[
m(\psi|x) = \int_{R^p} \psi(\theta) f(||\theta - x||) d\theta
\]
\[ M(\psi|x) = \frac{1}{C_f} \int_{\mathbb{R}^p} \psi(\theta) F(\|\theta - x\|) d\theta \]

where \( C_f = \left\{ \pi^{p/2}/\Gamma(p/2+1) \right\} \int_0^\infty z^{p+1} f(z) dz \). Notice that \( F(\cdot)/C_f \) is a probability density function because

\[
\int_{\mathbb{R}^p} \|y - \theta\|^\alpha F(\|y - \theta\|) dy = \int_{\mathbb{R}^p} \left\{ \int_0^\infty y^{\alpha - 1} f(s) ds \right\} dy = c_p \int_0^\infty r^{p+1+\alpha} \int_0^\infty s f(s) ds dr
\]

\[
= c_p \int_0^\infty r^{p+1+\alpha} \int_1^\infty t f(rt) dt dr
\]

\[
= c_p \int_1^\infty t \left\{ \int_0^\infty r^{p+1+\alpha} f(rt) dr \right\} dt
\]

\[
= c_p \int_1^\infty t^{-p-\alpha} \int_0^\infty z^{p+1+\alpha} f(z) dz
\]

\[
= \frac{c_p}{p + \alpha} \int_0^\infty z^{p+1+\alpha} f(z) dz.
\]

Then \( \delta_g \) is written as

\[
\delta_g(x) = x + C_f \frac{M(\nabla g|x)}{m(g|x)}.
\]

Now we state the main theorem of this paper.

**Theorem 3.1.** Assume \( F_1 \) with \( s > 5 \), \( G_1 - G_4 \) and \( FG_1 \). Then the gBayes estimator with respect to \( G(||\theta||) \) is admissible if \( \int_1^\infty r^{1-p} \{ G(r) \}^{-1} dr = \infty \).

**Proof of Theorem 3.1.** Let \( \delta_{g_1} \) denote the Bayes estimator with respect to the proper prior density \( g(\theta) h^2_1(\theta) = G(||\theta||) H^2_1(||\theta||) \) where \( H_1(\eta) \) has been given by (17) and let \( k(\eta) \) in (18) be \( \eta^{p-1} G(\eta) \). Then the Bayes risk difference of \( \delta_g \) and \( \delta_{g_1} \) with respect to the density \( g(\theta) h^2_1(\theta) \) is written as

\[
\Delta_i = \int_{\mathbb{R}^p} \left[ R(\theta, \delta_g) - R(\theta, \delta_{g_1}) \right] g(\theta) h^2_1(\theta) d\theta
\]

\[
= \int_{\mathbb{R}^p} \left\{ ||\delta_g - \theta||^2 - ||\delta_{g_1} - \theta||^2 \right\} f(||x - \theta||) g(\theta) h^2_1(\theta) d\theta dx
\]

\[
= \int_{\mathbb{R}^p} \left\{ ||\delta_g||^2 - ||\delta_{g_1}||^2 \right\} \int_{\mathbb{R}^p} f(||x - \theta||) g(\theta) h^2_1(\theta) d\theta d\theta
\]

\[
- 2(\delta_g - \delta_{g_1})' \int_{\mathbb{R}^p} \theta f(||x - \theta||) g(\theta) h^2_1(\theta) d\theta dx
\]
As in Brown and Hwang (1982), we have
\[
\Delta_i \leq 2C_j \int_{R^p} \left\| \frac{M(g\nabla h_i^2|x)}{m(gh_i^2|x)} \right\|^2 \frac{m(gh_i^2|x)dx}{m(gh_i^2|x)} \]
\[
+ 2C_j \int_{R^p} \left\| \frac{M(g\nabla h_i^2|x)}{m(g|x)} - \frac{M(g\nabla h_i^2|x)}{m(gh_i^2|x)} \right\|^2 \frac{m(gh_i^2|x)dx}{m(gh_i^2|x)} \]
\[
= 2C_j^2 (A_i + B_i) \quad \text{(say).}
\]

Using the Cauchy-Schwartz inequality for \(A_i\), we have
\[
A_i = 4 \int_{R^p} \|M(gh_i\nabla h_i|x)\|^2 \{m(gh_i^2|x)\}^{-1} dx
\]
\[
\leq 4 \int_{R^p} \frac{M(gh_i^2|x)}{m(gh_i^2|x)} M(\|\nabla h_i\|^2|x) dx.
\]

By Theorem A.2 in Appendix, there exists \(L_1\) such that
\[
M(gh_i^2|x)/m(gh_i^2|x) < L_1
\]
for all \(x\), all \(i\) and \(s > 5\). Then
\[
A_i \leq 4L_1 \int_{R^p} M(\|\nabla h_i\|^2|x) dx
\]
\[
= 4L_1 \int_{R^p} \{1/C_j\} F(\|x - \theta\|) dx \int_{R^p} g(\theta)\|\nabla h_i(\theta)\|^2 d\theta
\]
\[
= 4L_1 \int_{R^p} g(\theta)\|\nabla h_i(\theta)\|^2 d\theta
\]
\[
= 8L_1 \frac{\pi^{p/2}}{\Gamma(p/2)} \int_0^\infty t^{p-1} G(t) \left\{ \frac{d}{dt} H_i(t) \right\}^2 dt,
\]
which goes to 0 as \(i \to \infty\) by Theorem 2.3.

Next we consider \(B_i\). \(M(\nabla g|x)\) and \(M(\nabla gh_i^2|x)\) at \(x = 0\) are zero vectors because \(g\) and \(h_i^2\) are function of \(\|\theta\|\). So the integrand of \(B_i\) is bounded around \(x = 0\). When we
consider the asymptotic property of the integrand of $B_i$, note that there exists an $L_2$ such that $\eta|G'(\eta)/G(\eta)| \leq L_2$ for all $\eta > 0$ under the Assumption $G1$ because the regularly varying $G$ with index $\alpha$ satisfies $\lim_{\eta \to \infty} \eta G'(\eta)/G(\eta) = \alpha$. Then we have

\[
\left| \frac{M(\nabla_j g|x)}{m(g|x)} - \frac{M(\nabla_j gh^2_i|x)}{m(gh^2_i|x)} \right| \\
\leq \int_{R^p} \left| \frac{\theta_j}{\|\theta\|} G'(\|\theta\|) \left( \frac{1}{m(g|x)} - \frac{h_i^2}{m(gh^2_i|x)} \right) F(\|x - \theta\|) \right| d\theta \\
\leq \frac{2L_2}{\sqrt{m(g|x)m(gh^2_i|x)}} \int_{R^p} \frac{G(\|\theta\|)}{\|\theta\|} \left( \frac{1}{\sqrt{m(g|x)}} + \frac{h_i}{\sqrt{m(gh^2_i|x)}} \right) \left( \frac{1}{\sqrt{m(g|x)}} - \frac{h_i}{\sqrt{m(gh^2_i|x)}} \right) F(\|x - \theta\|) d\theta \\
< \frac{L_3}{\sqrt{m(gh^2_i|x)}} \sqrt{G(\|x\|)H_1(\|x\|)}
\]

for sufficiently large $\|x\|$, some constant $L_3$ and $s > 5$. Hence there exists $L_4$ and $L_5$ such that the integrand of $B_i$ is less than

$$\min\{L_4, L_5G(\|x\|)H_2^2(\|x\|)\}.$$ 

By (24) in Theorem 2.3 and the dominated convergence theorem, $B_i$ converges to 0 as $i \to \infty$.

\[\square\]

## A Appendix

### A.1 The Blyth method

There are several versions of the Blyth method. For our purpose, a following version from Brown (1971) and Brown and Hwang (1982) is useful.
Theorem A.1. Assume that there is an increasing sequence of proper densities such that 
\[ \int_{\|\theta\| \geq 1} g_1(\theta) d\theta > c \] 
for some positive \( c \) and \( \Delta_i \to 0 \) as \( i \to \infty \). Then \( \delta_g \) is admissible.

Proof. Suppose that \( \delta_g \) is inadmissible and let \( R(\theta, \delta') \leq R(\theta, \delta_g) \) for all \( \theta \) with strict inequality for some \( \theta \). Let \( \delta'' = (\delta_g + \delta')/2 \). Then, using Jensen’s inequality,

\[
R(\theta, \delta'') = \int \|\delta''(x) - \theta\|^2 f(\|x - \theta\|) dx 
< \left( \int \|\delta_g(x) - \theta\|^2 f(\|x - \theta\|) dx + \int \|\delta'(x) - \theta\|^2 f(\|x - \theta\|) dx \right) 
= \left[ R(\theta, \delta') + R(\theta, \delta_g) \right]/2 \leq R(\theta, \delta_g),
\]
for any \( \theta \). \( R(\theta, \delta'') \) and \( R(\theta, \delta_g) \) are both continuous functions of \( \theta \). Hence there exists an \( \epsilon > 0 \) such that \( R(\theta, \delta'') < R(\theta, \delta_g) - \epsilon \) for \( \|\theta\| \leq 1 \). Then

\[
\Delta_i \geq \int_{\|\theta\| \leq 1} [R(\theta, \delta_g) - R(\theta, \delta'')] g_1(\theta) d\theta 
\geq \int_{\|\theta\| \leq 1} [R(\theta, \delta_g) - R(\theta, \delta'')] g_1(\theta) d\theta 
\geq \epsilon \epsilon > 0,
\]
which contradicts \( \Delta_i \to 0 \). \( \square \)

A.2 The asymptotic behaviors of expected values

We give some results on the asymptotic behaviors of expected values when the location parameter diverges to infinity. Actually, in Section 3, we need an evaluation of the asymptotic behavior of expectation

\[
E_x[\rho(\theta)] = \int_{\mathbb{R}^p} \rho(\theta) f(\|\theta - x\|) d\theta
\]

for sufficiently large \( \|x\| \), where a random vector \( \theta \) has the density function \( f(\|\theta - x\|) \). This is the expected value with respect to the posterior distribution. Interchanging the roles of \( x \) and \( \theta \), in this appendix, we consider the asymptotic behavior of expectation

\[
E_\theta[\rho(X)] = \int_{\mathbb{R}^p} \rho(x) f(\|x - \theta\|) dx
\]

for sufficiently large \( \|\theta\| \), where a random vector \( X \) has the density function \( f(\|x - \theta\|) \).

Now we make the following regularity conditions on the density \( f \) and the function \( \rho \).
There exist $r_0 > 0$, $L > 0$, and $s > 1$, such that $r^{p+s}f(r) \leq L$ for all $r \geq r_0$.

$\rho(x)$ is written as $\rho(x) = \varrho(\|x\|)$, where $\varrho(r)$ is continuously differentiable in $r > 0$.

There exists $r_1 \geq 1$ and $t_1 \leq t_2$ such that $\varrho(r) > 0$ and $t_1 \leq r \varrho'(r)/\varrho(r) \leq t_2$ for all $r \geq r_1$.

The following lemma is useful. The proof, based on the integration of $(\log \varrho(r))' = \varrho'(r)/\varrho(r)$, is easy and omitted.

**Lemma A.1.** Under the assumption B2

\[
(z/y)^{t_1} \leq \varrho(z)/\varrho(y) \leq (z/y)^{t_2}
\]

for any $z > y \geq r_1$. Moreover

\[
\limsup_{y \to \infty} \sup_{\alpha y \leq z \leq \beta y} \varrho(z)/\varrho(y) \leq \max(\alpha^{t_1}, \beta^{t_2})
\]

for any $0 < \alpha < 1 < \beta$.

We now state the following theorem concerning the asymptotic behavior of $E[\rho(X)]$ for large $\|\theta\|$.

**Theorem A.2.** Assume F1, B1 and B2. If $s > \max(1, -t_1 - p, t_2)$ and $\int_0^1 r^{p-1} |\varrho(r)| dr < \infty$, then there exists $\epsilon > 0$ (say $\epsilon = \min(1, s + t_1 + p)/4$) such that

\[
\|\theta\|^\epsilon |E[\rho(X)] - \rho(\theta)| < C\varrho(\theta)
\]

for $\|\theta\| \geq 2\max(r_0, r_1)$. Moreover $C$ depends on $\rho$ (or $\varrho$) only through $r_1$, $t_1$, $t_2$ and $\{\varrho(r_1)\}^{-1} \int_0^1 r^{p-1} |\varrho(r)| dr$.

**Proof.** Fix $0 < \nu < 1$ (set $\nu = 1/2$ finally). Define

\[
V_\nu = \{x : \|x - \theta\| \leq \nu\|\theta\|\}
\]

\[
V'_\nu = \{x : (1 - \nu)\|\theta\| \leq \|x\| \leq (1 + \nu)\|\theta\|\}.
\]

Clearly $V_\nu \subset V'_\nu$. Then

\[
\|\theta\|^\epsilon |E[\rho(X) - \rho(\theta)]|
\]

\[
\leq \|\theta\|^\epsilon \left( \int_{V_\nu} + \int_{V'_\nu} \right) |\rho(x) - \rho(\theta)| f(\|x - \theta\|) dx
\]
\[
\begin{align*}
\leq & \|\theta\|^\epsilon \int_{V_\nu} |\rho(x) - \rho(\theta)| f(\|x - \theta\|)dx \\
& + \|\theta\|^\epsilon \rho(\theta) \int_{V_\nu^c} f(\|x - \theta\|)dx + \|\theta\|^\epsilon \int_{V_\nu^c} |\rho(x)| f(\|x - \theta\|)dx \\
= & I_1 + I_2 + I_3 \quad \text{(say)}. \quad \text{(27)}
\end{align*}
\]

Consider the first integral \(I_1\). If \(s > 1\), then \(m_1 = \int_{V_\nu} \|x - \theta\| f(\|x - \theta\|)dx\) is finite. Therefore for \(\|\theta\| \geq (1 - \nu)^{-1} r_1\) we have

\[
\begin{align*}
\|\theta\|^\epsilon \int_{V_\nu} |\rho(x) - \rho(\theta)| f(\|x - \theta\|)dx \\
& = \|\theta\|^\epsilon \int_{V_\nu} |(x - \theta)' \nabla \rho(x')| f(\|x - \theta\|)dx, \ x^* \in V_\nu \\
& \leq m_1 \|\theta\|^\epsilon \sup_{x \in V_\nu} |\nabla \rho(x)| \\
& \leq m_1 \|\theta\|^\epsilon \sup_{x \in V_\nu} |g'(\|x\|)| \\
& \leq m_1 \|\theta\|^\epsilon - 1 \sup_{x \in V_\nu} \|\theta\| \sup_{x \in V_\nu} \frac{g(\|x\|)}{g(\|\theta\|)} \sup_{x \in V_\nu} \frac{\|x\| |g'(|x|)|}{g(|x|)} \times \rho(\theta) \\
& \leq \frac{m_1}{1 - \nu} \max \{(1 - \nu)^{t_1}, \{1 + \nu\}^{t_2}\} \max(|t_1|, |t_2|) \times \rho(\theta)
\end{align*}
\]

for \(0 < \epsilon < 1\). Therefore we have \(I_1 \leq C_1 \rho(\theta)\) for some \(C_1\).

Now we consider the integral outside of \(V_\nu\). We only consider \(\|\theta\| \geq \max(\nu^{-1} r_0, r_1)\). Then for \(x \in V_\nu^c\)

\[
\|x - \theta\| \geq \nu \|\theta\| \geq r_0.
\]

Therefore we have, for \(0 \leq \alpha < s\)

\[
\begin{align*}
\int_{V_\nu^c} \|x\|^\alpha f(\|x - \theta\|)dx & \leq \int_{V_\nu^c} \{\|x - \theta\| + \|\theta\|\}^\alpha f(\|x - \theta\|)dx \\
& \leq (1 + 1/\nu)^\alpha \int_{V_\nu^c} \|x - \theta\|^\alpha f(\|x - \theta\|)dx \\
& \leq (1 + 1/\nu)^\alpha c_\nu L \int_{\nu \|\theta\|}^{\infty} r^{-s + \alpha - 1}dr \\
& = (1 + 1/\nu)^\alpha c_\nu L (\nu \|\theta\|-s + \alpha)\frac{1}{s - \alpha} \\
& \leq C_2(\alpha) \|\theta\|^\alpha - s, \quad \text{(28)}
\end{align*}
\]

where \(C_2(\alpha) = (1 + 1/\nu)^\alpha c_\nu L(\nu^{-s} - s)^{-1}\). Hence for the second term \(I_2\), if \(s > 1\) and \(0 < \epsilon < 1\), then \(I_2 \leq C_2(0) \rho(\theta)\).
We have seen that \( I_1 \) and \( I_2 \) are bounded from above assuming only \( s > 1 \). The third term \( I_3 \) of (27) is more problematic. Write

\[
I_3 = \|\theta\|^s \int_{V_\varphi^C} |\rho(x)||f(x - \theta) dx
\]

\[
\leq \|\theta\|^s \left( \int_{V_\varphi^C \cap \{\|x\| < r_1\}} + \int_{V_\varphi^C \cap \{r_1 \leq \|x\| \leq \|\theta\|\}} + \int_{V_\varphi^C \cap \{\|x\| > \|\theta\|\}} \right) |\rho(x)||f(x - \theta) dx
\]

\[
= I_{31} + I_{32} + I_{33} \quad \text{(say)}
\]

We take care of \( I_{33} \) first. Since \( g(r)r^{-t_2} \) is monotone nonincreasing for \( r \geq r_1, \rho(x)||x||^{-t_2} \leq \rho(\theta)||\theta||^{-t_2} \) for \( ||x|| > ||\theta||(\geq r_1) \). Therefore we have,

\[
I_{33} \leq \|\theta\|^{-t_2} \rho(\theta) \int_{V_\varphi^C \cap \{\|x\| > \|\theta\|\}} \|x\|^{t_2} f(||x - \theta||) dx.
\]

If \( 0 \leq t_2 < s \), as in (28)

\[
\int_{V_\varphi^C \cap \{\|x\| > \|\theta\|\}} \|x\|^{t_2} f(||x - \theta||) dx \leq \int_{V_\varphi^C} \|x\|^{t_2} f(||x - \theta||) dx
\]

\[
\leq C_2(t_2)||\theta||^{t_2 - s}
\]

and if \( t_2 < 0 \),

\[
\int_{V_\varphi^C \cap \{\|x\| > \|\theta\|\}} \|x\|^{t_2} f(||x - \theta||) dx \leq \|\theta\|^{t_2} \int_{V_\varphi^C} f(||x - \theta||) dx
\]

\[
\leq C_2(0)||\theta||^{t_2 - s}.
\]

Hence \( I_{33} \leq C_{33}\rho(\theta) \) where \( C_{33} = \max(C_2(t_2), C_2(0)) \).

Next we consider \( I_{31} \). For \( ||\theta|| \geq \max(\nu^{-1}r_0, r_1) \) and \( x \in V_\varphi^C \)

\[
f(||x - \theta||) \leq L||x - \theta||^{-p-s} \leq L(\nu||\theta||)^{-p-s}. \quad (29)
\]

Therefore

\[
I_{31} \leq \|\theta\|^s \nu^{-p-s}\||\theta||^{-p-s} \int_{||x|| \leq r_1} \rho(x) dx.
\]

Note that by simple change of variables we have

\[
\frac{\partial}{\partial r} \int_{||x|| \leq r} dx = c_p r^{p-1}.
\]
Then
$$\int_{|x| \leq r_1} |\rho(x)|\,dx = c_p \int_0^{r_1} r^{p-1}|\varrho(r)|\,dr.$$  

Therefore
$$I_{31} \leq C_s \|\theta\|^{\epsilon-p-s} \int_0^{r_1} r^{p-1}|\varrho(r)|\,dr;$$
where $C_s = L\nu^{-p-s}c_p$. On the other hand for $\|\theta\| \geq r_1$, $\rho(\theta) = \varrho(\|\theta\|)$ is bounded from below as
$$\varrho(r_1) r_1^{-t_1} \|\theta\|^{t_1} \leq \varrho(\|\theta\|).$$

Therefore
$$I_{31} \leq \|\theta\|^{\epsilon-p-s-t_1} \times C_s \frac{r_1^{t_1}}{\varrho(r_1)} \int_0^{r_1} r^{p-1}|\varrho(r)|\,dr \times \rho(\theta).$$

Hence if $s > -t_1 - p$, then we can choose $\epsilon > 0$ (say $\epsilon = (p + s + t_1)/4$) such that $\epsilon - p - s - t_1 < 0$ and hence $I_{31} \leq C_{31} \rho(\theta)$ where
$$C_{31} = C_s \frac{r_1^{t_1}}{\varrho(r_1)} \int_0^{r_1} r^{p-1}|\varrho(r)|\,dr.$$

Finally we consider $I_{32}$. Note $\varrho(r) \leq \varrho(\|\theta\|)\|\theta\|^{-t_1} r_1^{t_1}$ for $r_1 \leq r \leq \|\theta\|$ and (29). Then
$$I_{32} \leq \|\theta\|^{\epsilon-t_1} L\nu^{-p-s} \|\theta\|^{-p-s} \varrho(\|\theta\|) \int_{r_1 \leq |x| \leq \|\theta\|} \|x|^{t_1}\,dx \leq \|\theta\|^{\epsilon-p-s-t_1} \times C_s \int_{r_1}^{\|\theta\|} r^{p+t_1-1}\,dr \times \rho(\theta).$$

Consider the integral $Q = \int_{r_1}^{\|\theta\|} r^{p+t_1-1}\,dr$. If $p + t_1 < 0$, then
$$Q \leq r_1^{t_1+p}/(-t_1 - p).$$

Therefore as in the case of $I_{31}$, if $s > -t_1 - p$, then we can choose $\epsilon > 0$ (say $\epsilon = (p + s + t_1)/4$) such that $\epsilon - p - s - t_1 < 0$ and hence
$$I_{32} \leq r_1^{\epsilon-s} C_{32} \rho(\theta) \leq C_{32} \rho(\theta)$$
where $C_{32} = \{-1/(p + t_1)\} C_s$. If $p + t_1 \geq 0$,
$$Q = \int_{r_1}^{\|\theta\|} r^{p+t_1+\epsilon_3-1-\epsilon_3}\,dr \leq \frac{\|\theta\|^{p+t_1+\epsilon_3} r_1^{-\epsilon_3}}{p + t_1 + \epsilon_3}$$
for any $\epsilon_3 > 0$. Hence

$$I_{32} \leq \frac{C_*}{p + t_1 + \epsilon_3} \|\theta\|^{c + \epsilon_3 - s} \rho(\theta).$$

If $s > 1$, we can choose $\epsilon$ and $\epsilon_3$ (say $\epsilon = \epsilon_3 = 1/4$) such that $\epsilon + \epsilon_3 - s < 0$ and hence

$$I_{32} \leq C_{32} \rho(\theta),$$

where $C_{32} = (p + t_1 + \epsilon_3)^{-1} C_*.$

We have now confirmed that if $s > \max(1, -t_1 - p, t_2)$, there exist $\epsilon > 0$ and $C = C_1 + C_2 + C_{31} + C_{32} + C_{33}$, such that (26) folds for $\|\theta\| \geq \max(\nu^{-1} r_0, r_1, (1 - \nu)^{-1} r_1)$ which equals to $2 \max(r_0, r_1)$ for $\nu = 1/2$.

In Section 3, we also need asymptotic behavior of the expectation of $\rho(X) \times h_i^\gamma(X)$ where $h_i(\theta) = H_i(\|\theta\|)$ given by (17) and $\gamma > 0$.

**Corollary A.1.** Assume F1, B1 and B2. If $s > \max(1, \gamma - t_1 - p, t_2)$ and $\int_0^1 r^{p-1} |\rho(r)| dr < \infty$, there exists $\epsilon > 0$ (say $\epsilon = \min(1, s + t_1 + p - \gamma)/4$) such that

$$\|\theta\|^c |E_\theta[\rho(X)h_i^\gamma(X)] - \rho(\theta)h_i^\gamma(\theta)| < C \rho(\theta)h_i^\gamma(\theta)$$

(30)

for $\|\theta\| \geq 2d_1 \max(r_0, r_1, \eta_0)$. Moreover $C$ does not depend on $i$.

**Proof.** Since we have

$$\eta \{\varrho(\eta)H_i^\gamma(\eta)\}'/\{\varrho(\eta)H_i^\gamma(\eta)\} = \eta \varrho'(\eta)/\varrho(\eta) + \gamma \eta H_i^\gamma(\eta)/H_i(\eta),$$

under Assumption B2 and by part V of Theorem 2.1, for $\epsilon_1 = (s + t_1 + p - \gamma)/16(> 0)$ there exists $\eta_0$ such that

$$t_1 - \gamma - \gamma \epsilon_1 \leq \eta \{\varrho(\eta)H_i^\gamma(\eta)\}'/\{\varrho(\eta)H_i^\gamma(\eta)\} \leq t_2$$

for all $\eta \geq \max(\eta_0, r_1)$. Then for $\epsilon = \min(1, s + t_1 + p - \gamma)/4(> 0)$, (30) follows from Theorem A.2.

For $\lambda_1 = \max(\eta_0, r_1),$

$$\frac{1}{H_i^\gamma(\lambda_1) \varrho(\lambda_1)} \int_0^{\lambda_1} r^{p-1} |H_i^\gamma(r)\varrho(r)| dr \leq \frac{1}{H_i^\gamma(\lambda_1) \varrho(\lambda_1)} \int_0^{\lambda_1} r^{p-1} |\varrho(r)| dr$$

which implies that $C$ does not depend on $i$.  \qed
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