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SUPPLEMENT TO "EXPONENTIAL ERGODICITY OF THE BOUNCY PARTICLE SAMPLER"

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1. Proofs of Auxiliary results of Section 4.

Proof of Lemma 1. We prove invariance using the approach developed in [1], see also [2], where a link is provided between the invariant measures of $\{Z_t : t \geq 0\}$ and those of the embedded discrete-time Markov chain $\{\Theta_k : k \geq 0\} := \{(X_{\tau_k}, V_{\tau_k}) : k \geq 0\}$. The Markov transition kernel of this chain is given for $A \times B \in \mathcal{B}(\mathcal{Z})$ by

$$\begin{aligned}&\mathcal{Q}\left((x,v), A\times B\right)\\&= \int_0^\infty \exp\{-\int_0^s \bar\lambda(x+uv,v)\,\mathrm{d}u\}\bar\lambda(x+sv,v)K\left((x+sv,v),A\times B\right)\,\mathrm{d}s,\end{aligned}$$

where K is defined in equation (2.7) of the main manuscript. We also define for $A \times B \in \mathcal{B}(\mathcal{Z})$ the measure

$$\begin{aligned}\mu(A\times B) &:= \int \bar\lambda(x,v)\pi(\,\mathrm{d}x,\,\mathrm{d}v)K\left((x,v),A\times B\right)\\&= \int_{A\times B}\left[\Lambda_{\mathrm{ref}}(x)+\lambda(x,R(x)v)\right]\pi(\,\mathrm{d}x,\,\mathrm{d}v)\\&= \int_{A\times B}\bar\lambda(x,-v)\pi(\,\mathrm{d}x,\,\mathrm{d}v),\end{aligned}$$

as $\lambda(x, R(x)v) = \lambda(x, -v)$. This measure is finite by the integrability condition (A1). We set $\xi := (\mu(\mathcal{Z}))^{-1}$ and $\bar\mu := \xi\mu$. The measure $\bar\mu$ satisfies $\bar\mu = \mathcal{T}\pi$, where $\mathcal{T}$ is operator defined in [1, Section 3.3] mapping invariant measures of $\{Z_t : t \geq 0\}$ to invariant measures of $\{\Theta_k : k \geq 0\}$. By [1, Theorem 3], $\mathcal{T}$ is invertible. Therefore, from [1, Theorem 2], it suffices to prove the result to show that μ is invariant for $\{\Theta_k\}$ which we now establish.

For continuous, bounded $f : \mathcal{Z} \to \mathbb{R}$ we have

$$\begin{aligned}&\zeta \iiiint \mu(\,\mathrm{d}x,\,\mathrm{d}v)\mathcal{Q}\left((x,v),\,\mathrm{d}y,\,\mathrm{d}w\right)f(y,w)\\&= \iint \mathrm{e}^{-U(x)}\,\mathrm{d}x\psi(\,\mathrm{d}v)\bar\lambda(x,-v)\end{aligned}$$

$$
\begin{aligned}
&\times \int_0^\infty \exp\{-\int_0^s \bar\lambda(x+uv,v)\,\mathrm{d}u\}\bar\lambda(x+sv,v)Kf(x+sv,v)\,\mathrm{d}s\\
&= \int_{s=0}^\infty \mathrm{d}s \iint \mathrm{e}^{-U(x)}\,\mathrm{d}x\psi(\,\mathrm{d}v)\bar\lambda(x,-v)\\
&\quad \times \exp\{-\int_0^s \bar\lambda(x+uv,v)\,\mathrm{d}u\}\bar\lambda(x+sv,v)Kf(x+sv,v)
\end{aligned}
$$

and letting $z = x + sv$

$$
\begin{aligned}
&= \int_{s=0}^\infty \mathrm{d}s \iint \,\mathrm{d}z\psi(\,\mathrm{d}v)\mathrm{e}^{-U(z-sv)}\bar\lambda(z-sv,-v)\\
&\quad \times \exp\{-\int_0^s \bar\lambda(z+(u-s)v,v)\,\mathrm{d}u\}\bar\lambda(z,v)Kf(z,v)\\
&= \int_{s=0}^\infty \mathrm{d}s \iint \,\mathrm{d}z\psi(\,\mathrm{d}v)\bar\lambda(z-sv,-v)\\
&\quad \times \exp\{-U(z-sv) - \int_0^s \bar\lambda(z-wv,v)\,\mathrm{d}w\}\bar\lambda(z,v)Kf(z,v).
\end{aligned}
$$

Since $t \mapsto U(x+tv)$ is absolutely continuous, we can write

$$
\begin{aligned}
U(z) &= U(z-sv) + \int_{w=0}^s \langle \nabla U(z-wv), v\rangle\,\mathrm{d}w\\
&= U(z-sv) + \int_{w=0}^s \big[\max\{\langle \nabla U(z-wv), v\rangle, 0\}\\
&\qquad + \min\{\langle \nabla U(z-wv), v\rangle, 0\}\big]\,\mathrm{d}w,
\end{aligned}
$$

and it follows that

$$
\begin{aligned}
&U(z) + \int_{w=0}^s \max\{\langle \nabla U(z-wv), -v\rangle, 0\}\,\mathrm{d}w\\
&\quad = U(z-sv) + \int_{w=0}^s \max\{\langle \nabla U(z-wv), v\rangle, 0\}\,\mathrm{d}w.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\zeta \iiiint \mu(\,\mathrm{d}x,\,\mathrm{d}v)\mathcal{Q}\left((x,v),\,\mathrm{d}y,\,\mathrm{d}w\right)f(y,w)\\
&= \int_{s=0}^\infty \mathrm{d}s \iint \,\mathrm{d}z\psi(\,\mathrm{d}v)\bar\lambda(z-sv,-v)\\
&\quad \times \exp\{-U(z) - \int_0^s \bar\lambda(z-wv,-v)\,\mathrm{d}w\}\bar\lambda(z,v)Kf(z,v)\\
&= \iint \mathrm{e}^{-U(z)}\,\mathrm{d}z\psi(\,\mathrm{d}v)\bar\lambda(z,v)Kf(z,v)\\
&\quad \times \int_{s=0}^\infty \mathrm{d}s\bar\lambda(z-sv,-v)\exp\{-\int_0^s \bar\lambda(z-wv,-v)\,\mathrm{d}w\}
\end{aligned}
$$

$$
\begin{aligned}
&= \iint \mathrm{e}^{-U(z)} \,\mathrm{d}z \psi(\,\mathrm{d}v) \bar{\lambda}(z,v) K f(z,v) \\
&= \zeta \iint \pi(\,\mathrm{d}z,\,\mathrm{d}v) \bar{\lambda}(z,v) K f(z,v) = \zeta \iint \mu(\,\mathrm{d}z,\,\mathrm{d}v) f(z,v),
\end{aligned}
$$

proving that μ is invariant for $\mathcal{Q}$. □

Proof of Lemma 2. The proof is inspired by [5]. Let $f : B(0, T/6) \times \mathbb{S}^{d-1} \to [0,\infty)$ be a bounded non-negative function. Let E be the event that exactly two events have occurred up to time T, and both of them are refreshments. Then

$$
\begin{aligned}
&\mathbb{E}^z[f(Z_T)] \\
&\geq \mathbb{E}^z[f(Z_T)\mathbf{1}_E] \\
&= \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \psi(\,\mathrm{d}v_1)\psi(\,\mathrm{d}v_2) \int_{t=0}^{T} \int_{s=0}^{T-t} \,\mathrm{d}s \,\mathrm{d}t \bar{\lambda}(x_0 + t v_0, v_0) \\
&\quad \times \exp\left\{ -\int_{u=0}^{t} \bar{\lambda}(x_0 + u v_0, v_0) \,\mathrm{d}u \right\} \frac{\Lambda_{\mathrm{ref}}(x + t v_0)}{\bar{\lambda}(x + t v_0, v_0)} \bar{\lambda}(x_0 + t v_0 + s v_1, v_1) \\
&\quad \times \exp\left\{ -\int_{w=0}^{s} \bar{\lambda}(x_0 + v_0 t + w v_1, v_1) \,\mathrm{d}w \right\} \frac{\Lambda_{\mathrm{ref}}(x + t v_0 + s v_1)}{\bar{\lambda}(x + t v_0 + s v_1, v_1)} \\
&\quad \times \exp\left\{ -\int_{r=0}^{T-s-t} \bar{\lambda}(x_0 + v_0 t + s v_1 + r v_2, v_2) \,\mathrm{d}r \right\} \\
&\quad \times f\left(x_0 + t v_0 + s v_1 + (T-s-t) v_2, v_2\right) \\
&= \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \psi(\,\mathrm{d}v_1)\psi(\,\mathrm{d}v_2) \int_{t=0}^{T} \int_{s=0}^{T-t} \,\mathrm{d}s \,\mathrm{d}t \; \Lambda_{\mathrm{ref}}(x + t v_0) \\
&\quad \times \exp\left\{ -\int_{u=0}^{t} \bar{\lambda}(x_0 + u v_0, v_0) \,\mathrm{d}u \right\} \Lambda_{\mathrm{ref}}(x + t v_0 + s v_1) \\
&\quad \times \exp\left\{ -\int_{w=0}^{s} \bar{\lambda}(x_0 + v_0 t + w v_1, v_1) \,\mathrm{d}w \right\} \\
&\quad \times \exp\left\{ -\int_{r=0}^{T-s-t} \bar{\lambda}(x_0 + v_0 t + s v_1 + r v_2, v_2) \,\mathrm{d}r \right\} \\
&\quad \times f\left(x_0 + t v_0 + s v_1 + (T-s-t) v_2, v_2\right).
\end{aligned}
$$

Since the process moves at unit speed and $|x_0| \leq T/6$, it follows that $\sup_{t \leq T} |X_t| \leq 7T/6$. Let

$$
K := \sup\left\{ \bar{\lambda}(x,v) : |x| \leq 7T/6,\, v \in \mathbb{S}^{d-1} \right\} < \infty,
$$

and recall that $\bar{\lambda}(x,v) \geq \lambda_{\mathrm{ref}} > 0$. Therefore

$$
\begin{aligned}
&\mathbb{E}^z[f(Z_T)] \\
&\geq \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \psi(\,\mathrm{d}v_1)\psi(\,\mathrm{d}v_2) \int_{s=0}^{T} \int_{t=0}^{T-s} \mathrm{d}s\,\mathrm{d}t\ \lambda_{\mathrm{ref}}\lambda_{\mathrm{ref}} \\
&\qquad \times \exp\left\{-\int_{u=0}^{t} K\,\mathrm{d}u - \int_{w=0}^{s} K\,\mathrm{d}w - \int_{r=0}^{T-s-t} K\,\mathrm{d}r\right\} \\
&\qquad \times f\left(x_0 + sv_0 + tv_1 + (T-s-t)v_2, v_2\right) \\
&= \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \psi(\,\mathrm{d}v_1)\psi(\,\mathrm{d}v_2) \int_{s=0}^{T} \int_{t=0}^{T-s} \mathrm{d}s\,\mathrm{d}t \lambda_{\mathrm{ref}}^2 \\
&\quad \times \exp\left\{-Kt - Ks - K(T-s-t)\right\} f\left(x_0 + sv_0 + tv_1 + (T-s-t)v_2, v_2\right) \\
&= \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \psi(\,\mathrm{d}v_1)\psi(\,\mathrm{d}v_2) \int_{s=0}^{T} \mathrm{d}s \int_{t=s}^{T} \mathrm{d}t \lambda_{\mathrm{ref}}^2 \exp\left\{-KT\right\} \\
&\qquad \times f\left(x_0 + sv_0 + (t-s)v_1 + (T-t)v_2, v_2\right) \\
&= \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \psi(\,\mathrm{d}v_1)\psi(\,\mathrm{d}v_2) \int_{t=0}^{T} \mathrm{d}t \\
&\qquad \times \int_{s=0}^{t} \mathrm{d}s \lambda_{\mathrm{ref}}^2 \exp\left\{-KT\right\} f\left(x_0 + sv_0 + (t-s)v_1 + (T-t)v_2, v_2\right), \\
&\geq \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \psi(\,\mathrm{d}v_1)\psi(\,\mathrm{d}v_2) \int_{t=5T/6}^{T} \mathrm{d}t \\
&\qquad \times \int_{s=0}^{t} \mathrm{d}s \lambda_{\mathrm{ref}}^2 \exp\left\{-KT\right\} f\left(x_0 + sv_0 + (t-s)v_1 + (T-t)v_2, v_2\right) \\
&= \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \psi(\,\mathrm{d}v_1)\psi(\,\mathrm{d}v_2) \int_{t=5T/6}^{T} \mathrm{d}t\ t \\
&\qquad \times \int_{r=0}^{1} \mathrm{d}r \lambda_{\mathrm{ref}}^2 \exp\left\{-KT\right\} f\left(x_0 + rtv_0 + (t-rt)v_1 + (T-t)v_2, v_2\right) \\
&\geq \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \psi(\,\mathrm{d}v_1)\psi(\,\mathrm{d}v_2) \frac{5T}{6} \int_{t=5T/6}^{T} \mathrm{d}t \\
&\qquad \times \int_{r=0}^{1} \mathrm{d}r \lambda_{\mathrm{ref}}^2 \exp\left\{-KT\right\} f\left(x_0 + trv_0 + t(1-r)v_1 + (T-t)v_2, v_2\right).
\end{aligned}
$$

Fix $t \in (5T/6, T]$ and $v_2 \in \mathbb{S}^{d-1}$ so that $x' := x_0 + (T-t)v_2$ is now fixed. Since $T \geq t > 5T/6$ it follows that $T - t < T/6$. Since also $|x_0| \leq T/6$ we must have that $|x'| \leq T/3$. Let $x'' \in B(0, T/6)$ be arbitrary. Then it follows that $|x' - x''| \leq T/2$.

We will now show that there exist $v_* \in \mathbb{S}^{d-1}$ and $r_* \in [0,1]$ such that

$$x' + tr_* v_0 + t(1-r_*)v_* = x''.$$

In the trivial case where $x'' = x' + trv_0$ for some $r \in [0,1]$, then since $t > |x'' - x'|$ it must hold that $r < 1$ and thus the representation is trivially satisfied with $v_* = -v_0$ and $r_* = r + (1-r)/2$.

If this is not the case, then for any $r \in [0,1]$ define $l(r) := rt + |x'' - x' - rtv_0| > rt$ and consider the path $f_r : [0, l(r)] \mapsto \mathbb{R}^d$ given by

$$f_r(s) = x' + sv_0, \text{ for } s \in [0, rt],$$
$$f_r(s) = x' + rtv_0 + (s - rt)\frac{x'' - x' - rtv_0}{|x'' - x' - rtv_0|}, \text{ for } s > rt.$$

For each $r \in [0,1]$ the path starts at x', ends at x'' and has length given by $l(r)$. By definition we have $l(0) = |x'' - x'|$ and since $t > |x'' - x'|$ it follows from the triangle inequality that $|x'' - x' - tv_0| \geq t - |x'' - x'| > 0$ and thus that $l(1) > t$. Since $l(\cdot)$ is continuous, by the intermediate value theorem there exists $r_* \in [0,1]$ such that $l(r_*) = t$, whence the definition of $l(\cdot)$ implies that

$$|x'' - x' - r_*tv_0| = (1 - r_*)t.$$

Letting

$$v_* := \frac{x'' - x' - r_*tv_0}{|x'' - x' - r_*tv_0|} \in \mathbb{S}^{d-1},$$

we have

$$x'' = x' + r_*tv_0 + t(1 - r_*)v_*.$$

To proceed we define the measure $\mu = \mu_{t,x_0,v_0}$ on $\mathbb{R}^d$ as

$$\mu(V) := \int_{\mathbb{S}^{d-1}} \psi(\,\mathrm{d}v_1) \int_{r=0}^{1} \mathrm{d}r\, \mathbb{1}_V \left(x' + rtv_0 + t(1-r)v_1\right),$$

for Borel sets $V \subset \mathbb{R}^d$. It is obvious that μ is absolutely continuous with respect to d-dimensional Lebesgue measure. Letting $R \sim U[0,1]$ and $V \sim \psi$ be independent, we have for any $\delta > 0$ and $x'' \in B(0, T/6)$

$$\begin{aligned}
\mu\left(B(x'', \delta)\right) &= \mathbb{P}\left\{|x' + tRv_0 + t(1-R)V - x''| \leq \delta\right\} \\
&= \mathbb{P}\left\{|tRv_0 + t(1-R)V - tr_*v_0 - t(1-r_*)v_*| \leq \delta\right\} \\
&= \mathbb{P}\left\{|t(R - r_*)v_0 + t(V - v_*) - t(RV - r_*v_*)| \leq \delta\right\} \\
&= \mathbb{P}\left\{|t(R - r_*)v_0 + t(V - v_*) - t(RV - Rv_* + Rv_* - r_*v_*)| \leq \delta\right\} \\
&= \mathbb{P}\left\{|t(R - r_*)v_0 + t(V - v_*) - tR(V - v_*) - t(R - r_*)v_*)| \leq \delta\right\} \\
&\geq \mathbb{P}\left\{T|R - r_*| + T|V - v_*| + T|V - v_*| + T|R - r_*| \leq \delta\right\} \\
&= \mathbb{P}\left\{|R - r_*| + |V - v_*| \leq \frac{\delta}{2T}\right\}
\end{aligned}$$

$$\geq \mathbb{P}\left\{\max\{|R-r_*|,|V-v_*|\} \leq \frac{\delta}{4T}\right\}$$
$$= \mathbb{P}\left\{|R-r_*| \leq \frac{\delta}{4T}\right\}\mathbb{P}\left\{|V-v_*| \leq \frac{\delta}{4T}\right\}$$
$$\geq \frac{\delta}{4T} \times \Big(C_1\Big(\frac{\delta}{4T}\Big)\Big)^{d-1} = C_2(T)\delta^d,$$

where $C_1 > 0$ is a constant, and where $C_i(\cdot)$ denotes quantities depending only on the variables in the bracket. Therefore, for all $t > 5T/6$ and $v_2 \in \mathbb{S}^{d-1}$, we can bound the density of μ with respect to Lebesgue measure, on the set $B(0,T/6)$, from below by a constant $C_3(T,d) > 0$

$$\int_{\mathbb{S}^{d-1}} \psi(\,\mathrm{d}v_1) \int_{r=0}^{1} \mathrm{d}r f\left(x_0 + rtv_0 + t(1-r)v_1 + (T-t)v_2, v_2\right)$$
$$\geq \int \mu(\,\mathrm{d}x'')f(x'',v_2) \geq C_3(T,d)\int_{B(0,T/6)} f\left(x'',v_2\right)\,\mathrm{d}x'',$$

and thus

$$\mathbb{E}^z[f(Z_T)] \geq C_4(T,d,\lambda_{\mathrm{ref}})\int_{\mathbb{S}^{d-1}}\psi(\,\mathrm{d}v_2)\int_{t=5T/6}^{T}\mathrm{d}t\int_{x''\in B(0,T/6)} f\left(x'',v_2\right)\,\mathrm{d}x''$$
$$\geq C_5(T,d,\lambda_{\mathrm{ref}})\int_{\mathbb{S}^{d-1}}\int_{x''\in B(0,T/6)} f\left(x'',v\right)\,\mathrm{d}x''\psi(\,\mathrm{d}v).$$

Since f is generic, we conclude that for all $z = (x'',v) \in B(0,T/6)\times\mathbb{S}^{d-1}$, and any Borel set $A \subseteq B(0,T/6)\times\mathbb{S}^{d-1}$

$$\mathbb{P}^z(Z_T \in A) \geq C_5(T,d,\lambda_{\mathrm{ref}})\iint_A \psi(\,\mathrm{d}v)\,\mathrm{d}x,$$

whence it follows that for any $R > 0$ the set $B(0,R)\times\mathbb{S}^{d-1}$ is petite.

Given any compact set $U \subset \mathbb{R}^d \times \mathbb{S}^{d-1}$, we can find $R > 0$ such that $U \subset B(0,R)\times\mathbb{S}^{d-1}$, and we can easily conclude using the above that U must also be petite.

Finally let $A \subseteq \mathcal{Z}$ such that $\iint_A \psi(\,\mathrm{d}v)\,\mathrm{d}x > 0$. We can find $R > 0$ such that the set $A' := A \cap (B(0,R)\times\mathbb{S}^{d-1})$ satisfies $\iint_{A'}\psi(\,\mathrm{d}v)\,\mathrm{d}x > 0$. Let $z = (x,v) \in \mathcal{Z}$ be arbitrary and for some fixed $\epsilon > 0$ define $T := \max\{6|x|+\epsilon, 6R+\epsilon\}$. Then $z \in B(0,T/6)\times\mathbb{S}^{d-1}$, $A' \subseteq B(0,T/6)\times\mathbb{S}^{d-1}$ and thus by the first part of the lemma

$$\mathbb{P}^z(Z_T \in A') \geq C_5(T,d,\lambda_{\mathrm{ref}})\iint_A \psi(\,\mathrm{d}v)\,\mathrm{d}x > 0.$$

Therefore since $A' \subseteq A$, writing $\tau_B := \inf\{t \geq 0 : Z_t \in B\}$ for a measurable set $B \subseteq \mathcal{Z}$, we have

$$\mathbb{P}^z\left\{\tau_A < \infty\right\} \geq \mathbb{P}^z\left\{\tau_{A'} < \infty\right\} \geq \mathbb{P}^z\left\{Z_T \in A'\right\} > 0.$$

Irreducibility then follows from [4, Proposition 2.1]. □

2. Proofs of results of Section 5.

PROOF OF LEMMA 4. That $V \in \mathcal{D}(\widetilde{\mathcal{L}})$ follows from the discussion in Section 5.1. We now establish that V is a Lyapunov function. First we compute $\widetilde{\mathcal{L}}V(x,v)$. Notice that if $\langle \nabla U(x), v\rangle \neq 0$, then by continuity there will be a neighbourhood of (x,v) on which $V(x,v)$ will be differentiable. Therefore at those points we will use equation (2.8) of the main manuscript.

Case $\langle \nabla U(x), v\rangle > 0$. We have

$$\langle \nabla V(x,v), v\rangle = \frac{1}{2}V(x,v)\langle \nabla U(x), v\rangle,$$

and adding the reflection part we obtain

$$\begin{aligned}
&\langle \nabla V(x,v), v\rangle + \langle \nabla U(x), v\rangle\left[V(x,R_x v) - V(x,v)\right]\\
&= \frac{1}{2}V(x,v)\langle \nabla U(x), v\rangle\\
&\qquad + \langle \nabla U(x), v\rangle\left[\frac{\mathrm{e}^{U(x)/2}}{\sqrt{\lambda_{\mathrm{ref}} + \langle \nabla U(x), -v\rangle_+}}\frac{\sqrt{\lambda_{\mathrm{ref}} + \langle \nabla U(x), -v\rangle_+}}{\sqrt{\lambda_{\mathrm{ref}} + \langle \nabla U(x), v\rangle}} - V(x,v)\right]\\
&= -\frac{1}{2}V(x,v)\langle \nabla U(x), v\rangle + \langle \nabla U(x), v\rangle V(x,v)\frac{\sqrt{\lambda_{\mathrm{ref}}}}{\sqrt{\lambda_{\mathrm{ref}} + \langle \nabla U(x), v\rangle}}.
\end{aligned}$$

The refreshment term is given by

$$\begin{aligned}
&\mathrm{e}^{U(x)/2}\lambda_{\mathrm{ref}}\int \psi(\,\mathrm{d}w)\left[\frac{1}{\sqrt{\lambda_{\mathrm{ref}} + \langle \nabla U(x), w\rangle_+}} - \frac{1}{\sqrt{\lambda_{\mathrm{ref}}}}\right]\\
&= \mathrm{e}^{U(x)/2}\lambda_{\mathrm{ref}}\int \psi(\,\mathrm{d}w)\frac{1}{\sqrt{\lambda_{\mathrm{ref}} + \langle \nabla U(x), w\rangle_+}} - \lambda_{\mathrm{ref}}V(x,v)\\
&= \mathrm{e}^{U(x)/2}\lambda_{\mathrm{ref}}\int_{\langle \nabla U(x), w\rangle > 0} \psi(\,\mathrm{d}w)\frac{1}{\sqrt{\lambda_{\mathrm{ref}} + \langle \nabla U(x), w\rangle_+}}\\
&\qquad + \mathrm{e}^{U(x)/2}\lambda_{\mathrm{ref}}\int_{\langle \nabla U(x), w\rangle \leq 0} \psi(\,\mathrm{d}w)\frac{1}{\sqrt{\lambda_{\mathrm{ref}}}} - \lambda_{\mathrm{ref}}V(x,v)\\
&= \mathrm{e}^{U(x)/2}\lambda_{\mathrm{ref}}\int_{\langle \nabla U(x), w\rangle > 0} \psi(\,\mathrm{d}w)\frac{1}{\sqrt{\lambda_{\mathrm{ref}} + \langle \nabla U(x), w\rangle_+}} - \frac{1}{2}\lambda_{\mathrm{ref}}V(x,v),
\end{aligned}$$

since $\psi\{w : \langle \nabla U(x), w\rangle > 0\} = 1/2$. Thus overall when $\langle \nabla U(x), v\rangle > 0$ we have

$$\widetilde{\mathcal{L}}V(x,v) = \frac{1}{2}V(x,v)\langle \nabla U, v\rangle + \langle \nabla U(x), v\rangle\left[V(x,R_x v) - V(x,v)\right]$$

$$
\begin{aligned}
&+ \mathrm{e}^{U(x)/2}\lambda_{\mathrm{ref}} \int_{\mathbb{S}^{d-1}} \psi(\,\mathrm{d}w) \left[\frac{1}{\sqrt{\lambda_{\mathrm{ref}} + \langle \nabla U(x), w\rangle_+}} - \frac{1}{\sqrt{\lambda_{\mathrm{ref}}}} \right] \\
&= -\frac{1}{2} V(x,v) \Bigg[\langle \nabla U(x), v\rangle - 2\frac{\langle \nabla U(x), v\rangle \sqrt{\lambda_{\mathrm{ref}}}}{\sqrt{\lambda_{\mathrm{ref}} + \langle \nabla U(x), v\rangle}} + \lambda_{\mathrm{ref}} \\
&\qquad - 2\lambda_{\mathrm{ref}}^{3/2} \int_{\langle \nabla U(x), w\rangle \geq 0} \frac{\psi(\,\mathrm{d}w)}{\sqrt{\lambda_{\mathrm{ref}} + \langle \nabla U(x), w\rangle}} \Bigg] \\
&= -\frac{1}{2} V(x,v) \Bigg[\langle \nabla U(x), v\rangle - 2\frac{\langle \nabla U(x), v\rangle \sqrt{\lambda_{\mathrm{ref}}}}{\sqrt{\lambda_{\mathrm{ref}} + \langle \nabla U(x), v\rangle}} + \lambda_{\mathrm{ref}} \\
&\qquad - 2\lambda_{\mathrm{ref}} \int_{\langle \nabla U(x), w\rangle \geq 0} \frac{\psi(\,\mathrm{d}w)}{\sqrt{1 + \frac{\langle \nabla U(x), w\rangle}{\lambda_{\mathrm{ref}}}}} \Bigg].
\end{aligned}
$$

Case $\langle \nabla U(x), v\rangle < 0$. In this case

$$
\langle \nabla V(x,v), v\rangle = -\frac{1}{2} V(x,v) \left[\langle \nabla U, -v\rangle - \frac{1}{\lambda_{\mathrm{ref}} + \langle \nabla U(x), -v\rangle} \langle v, \Delta U(x) v\rangle \right].
$$

Since $\langle \nabla U(x), v\rangle_+ = 0$ there is no reflection and thus overall

$$
\begin{aligned}
\widetilde{\mathcal{L}} V(x,v) &= -\frac{1}{2} V(x,v) \left[\langle \nabla U(x), -v\rangle - \frac{1}{\lambda_{\mathrm{ref}} + \langle \nabla U(x), -v\rangle} \langle v, \Delta U(x) v\rangle \right] \\
&\quad - \lambda_{\mathrm{ref}} V(x,v) + \frac{1}{2}\lambda_{\mathrm{ref}} \frac{\mathrm{e}^{U(x)/2}}{\sqrt{\lambda_{\mathrm{ref}}}} \\
&\quad + \lambda_{\mathrm{ref}} V(x,v) \int_{\langle \nabla U(x), w\rangle \geq 0} \psi(\,\mathrm{d}w) \frac{\sqrt{\lambda_{\mathrm{ref}} + \langle \nabla U(x), -v\rangle}}{\sqrt{\lambda_{\mathrm{ref}} + \langle \nabla U(x), w\rangle}} \\
&= -\frac{1}{2} V(x,v) \left[\langle \nabla U(x), -v\rangle - \frac{1}{\lambda_{\mathrm{ref}} + \langle \nabla U(x), -v\rangle} \langle v, \Delta U(x) v\rangle + 2\lambda_{\mathrm{ref}} \right] \\
&\quad + \frac{1}{2}\lambda_{\mathrm{ref}} V(x,v) \frac{\sqrt{\lambda_{\mathrm{ref}} + \langle \nabla U(x), -v\rangle}}{\sqrt{\lambda_{\mathrm{ref}}}} + \\
&\quad + \lambda_{\mathrm{ref}} V(x,v) \int_{\langle \nabla U(x), w\rangle \geq 0} \psi(\,\mathrm{d}w) \frac{\sqrt{\lambda_{\mathrm{ref}} + \langle \nabla U(x), -v\rangle}}{\sqrt{\lambda_{\mathrm{ref}} + \langle \nabla U(x), w\rangle}} \\
&= -\frac{1}{2} V(x,v) \Bigg[\langle \nabla U(x), -v\rangle + 2\lambda_{\mathrm{ref}} - \frac{1}{\lambda_{\mathrm{ref}} + \langle \nabla U(x), -v\rangle} \langle v, \Delta U(x) v\rangle \\
&\quad - \lambda_{\mathrm{ref}} \frac{\sqrt{\lambda_{\mathrm{ref}} + \langle \nabla U(x), -v\rangle}}{\sqrt{\lambda_{\mathrm{ref}}}} \\
&\qquad - 2\lambda_{\mathrm{ref}} \int_{\langle \nabla U(x), w\rangle \geq 0} \psi(\,\mathrm{d}w) \frac{\sqrt{\lambda_{\mathrm{ref}} + \langle \nabla U(x), -v\rangle}}{\sqrt{\lambda_{\mathrm{ref}} + \langle \nabla U(x), w\rangle}} \Bigg].
\end{aligned}
$$

Case $\langle \nabla U(x), v\rangle = 0$. In this case we compute $\widetilde{\mathcal{L}}V(x,v)$ as

$$\widetilde{\mathcal{L}}V(x,v) = \frac{\mathrm{d}}{\mathrm{d}t}V(x+tv,v)\Big|_{t=0+} + \lambda_{\text{ref}}\left[\int \psi(\,\mathrm{d}w)V(x,w) - V(x,v)\right], \tag{2.1}$$

since the reflection term vanishes.

We first compute the directional derivative for which we can distinguish two cases. Suppose first that $\langle \Delta U(x)v, -v\rangle > 0$. Then we have that for all $t > 0$ small enough

$$\langle \nabla U(x+tv), -v\rangle = 0 + t\langle \Delta U(x)v, -v\rangle + o(t) \geq 0.$$

Therefore, since $\langle \nabla U(x), v\rangle = 0$, in this case we can compute the first term of (2.1) as follows

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}V(x+tv,v)\Big|_{t=0+} \\
&= \lim_{t\to 0+}\frac{1}{t}\left[\frac{\exp(U(x+tv)/2)}{\sqrt{\lambda_{\text{ref}} + \langle \nabla U(x+tv), -v\rangle_+}} - \frac{\exp(U(x)/2)}{\sqrt{\lambda_{\text{ref}}}}\right] \\
&= \lim_{t\to 0+}\frac{1}{t}\left[\frac{\exp(U(x+tv)/2)}{\sqrt{\lambda_{\text{ref}} + \langle \nabla U(x+tv), -v\rangle}} - \frac{\exp(U(x)/2)}{\sqrt{\lambda_{\text{ref}}}}\right] \\
&= \lim_{t\to 0+}\frac{1}{t}\Big[\frac{\exp(U(x+tv)/2) - \exp(U(x)/2)}{\sqrt{\lambda_{\text{ref}} + \langle \nabla U(x+tv), -v\rangle}} \\
&\qquad + \exp(U(x)/2)\left(\frac{1}{\sqrt{\lambda_{\text{ref}} + \langle \nabla U(x+tv), -v\rangle}} - \frac{1}{\sqrt{\lambda_{\text{ref}}}}\right)\Big] \\
&= 0 - \frac{1}{2}\frac{\exp(U(x)/2)}{\sqrt{\lambda_{\text{ref}}}^3}\langle \Delta U(x)v, -v\rangle = -\frac{1}{2}V(x,v)\frac{\langle \Delta U(x)v, -v\rangle}{\lambda_{\text{ref}}}
\end{aligned}$$

Now consider the case where $\langle \Delta U(x)v, -v\rangle \leq 0$, then for all $t > 0$ small enough

$$\langle \nabla U(x+tv), -v\rangle = 0 + t\langle \Delta U(x)v, -v\rangle + o(t) \leq 0,$$

and therefore

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}V(x+tv,v)\Big|_{t=0+} \\
&= \lim_{t\to 0+}\frac{1}{t}\left[\frac{\exp(U(x+tv)/2)}{\sqrt{\lambda_{\text{ref}} + \langle \nabla U(x+tv), -v\rangle_+}} - \frac{\exp(U(x)/2)}{\sqrt{\lambda_{\text{ref}}}}\right] \\
&= \lim_{t\to 0+}\frac{1}{t}\left[\frac{\exp(U(x+tv)/2)}{\sqrt{\lambda_{\text{ref}} + 0}} - \frac{\exp(U(x)/2)}{\sqrt{\lambda_{\text{ref}}}}\right] = 0.
\end{aligned}$$

Overall we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x+tv,v)\Big|_{t=0+} = -\frac{1}{2}V(x,v)\langle \Delta U(x)v, -v\rangle_+.$$

Adding the refreshment term we find that in this case

$$\begin{aligned}\widetilde{\mathcal{L}}V(x,v) = -\frac{1}{2}V(x,v)\Big[&\frac{\langle \Delta U(x)v,-v\rangle_+}{\lambda_{\mathrm{ref}}} + \lambda_{\mathrm{ref}} \\ &- 2\lambda_{\mathrm{ref}}\int_{\langle \nabla U(x), w\rangle \geq 0} \frac{\psi(\,\mathrm{d}w)}{\sqrt{1+\frac{\langle \nabla U(x),w\rangle}{\lambda_{\mathrm{ref}}}}}\Big].\end{aligned}$$

Combining the three cases we obtain

(2.2)

$$2\frac{\widetilde{\mathcal{L}}V(x,v)}{V(x,v)} = \begin{cases} -\Big[\frac{\langle \Delta U(x)v,-v\rangle_+}{\lambda_{\mathrm{ref}}} + \lambda_{\mathrm{ref}} \\ \qquad -2\lambda_{\mathrm{ref}}\int_{\langle \nabla U(x),w\rangle\geq 0}\frac{\sqrt{\lambda_{\mathrm{ref}}}\psi(\,\mathrm{d}w)}{\sqrt{\lambda_{\mathrm{ref}}+\langle \nabla U(x),w\rangle}}\Big] & \langle \nabla U(x), v\rangle = 0, \\ -\Big[\langle \nabla U(x), v\rangle - 2\frac{\langle \nabla U(x),v\rangle\sqrt{\lambda_{\mathrm{ref}}}}{\sqrt{\lambda_{\mathrm{ref}}+\langle \nabla U(x),v\rangle}} + \lambda_{\mathrm{ref}} \\ \qquad -2\lambda_{\mathrm{ref}}\int_{\langle \nabla U(x),w\rangle\geq 0}\frac{\sqrt{\lambda_{\mathrm{ref}}}\psi(\,\mathrm{d}w)}{\sqrt{\lambda_{\mathrm{ref}}+\langle \nabla U(x),w\rangle}}\Big], & \langle \nabla U(x), v\rangle > 0, \\ -\Big[\langle \nabla U(x), -v\rangle + 2\lambda_{\mathrm{ref}} - \frac{1}{\lambda_{\mathrm{ref}}+\langle \nabla U(x),-v\rangle}\langle v, \Delta U(x) v\rangle \\ \quad -\lambda_{\mathrm{ref}}\frac{\sqrt{\lambda_{\mathrm{ref}}+\langle \nabla U(x),-v\rangle}}{\sqrt{\lambda_{\mathrm{ref}}}} \\ \qquad -2\lambda_{\mathrm{ref}}\int_{\langle \nabla U(x),w\rangle\geq 0}\psi(\,\mathrm{d}w)\frac{\sqrt{\lambda_{\mathrm{ref}}+\langle \nabla U(x),-v\rangle}}{\sqrt{\lambda_{\mathrm{ref}}+\langle \nabla U(x),w\rangle}}\Big], & \langle \nabla U(x), v\rangle < 0. \end{cases}$$

*Condition (*A*).* We have that $\overline{\lim}_{|x|\to\infty}\|\Delta U(x)\| \leq \alpha_1$ and $\underline{\lim}_{|x|\to\infty}|\nabla U(x)| = \infty$. Since $\lambda_{\mathrm{ref}} > 0$ and $1/\sqrt{\cos(\theta)} \in L^1([0,\pi/2], \mathrm{d}\theta)$, for any $\epsilon > 0$ we can find $K > 0$ such that for all $|x| > K$

$$\int_{\langle \nabla U(x),w\rangle\geq 0}\frac{\psi(\,\mathrm{d}w)}{\sqrt{\lambda_{\mathrm{ref}}+\langle \nabla U(x),w\rangle}} \leq \int_{\theta=0}^{\pi/2}\frac{p_\vartheta(\theta)\,\mathrm{d}\theta}{\sqrt{|\nabla U(x)|\cos(\theta)}} \leq \frac{\epsilon}{\sqrt{\lambda_{\mathrm{ref}}}}, \tag{2.3}$$

Case $\langle \nabla U(x), v\rangle = 0$. Suppose that $|x| > K$. Then from (2.2), by dropping the first term which is negative,

$$\begin{aligned}2\frac{\widetilde{\mathcal{L}}V(x,v)}{V(x,v)} &\leq -\left[\lambda_{\mathrm{ref}} - 2\lambda_{\mathrm{ref}}\int_{\langle \nabla U(x),w\rangle\geq 0}\frac{\sqrt{\lambda_{\mathrm{ref}}}\psi(\,\mathrm{d}w)}{\sqrt{\lambda_{\mathrm{ref}}+\langle \nabla U(x),w\rangle}}\right] \\ &\leq -\lambda_{\mathrm{ref}}(1-2\epsilon).\end{aligned}$$

Case $\langle \nabla U(x), v\rangle > 0$. Again let $|x| > K$. From (2.2)

$$2\frac{\widetilde{\mathcal{L}}V(x,v)}{V(x,v)} \leq -\left[\langle \nabla U(x), v\rangle - 2\frac{\langle \nabla U(x), v\rangle \sqrt{\lambda_{\text{ref}}}}{\sqrt{\lambda_{\text{ref}} + \langle \nabla U(x), v\rangle}} + \lambda_{\text{ref}}(1-2\epsilon)\right].$$

For $w > 0$ consider the function

$$w - 2\frac{w\sqrt{\lambda_{\text{ref}}}}{\sqrt{\lambda_{\text{ref}} + w}} + \lambda_{\text{ref}}(1-2\epsilon).$$

Since for $w > 0$, $\lambda_{\text{ref}} + w \geq 2\sqrt{w}\sqrt{\lambda_{\text{ref}}}$ we have that

$$w - 2\frac{w\sqrt{\lambda_{\text{ref}}}}{\sqrt{\lambda_{\text{ref}} + w}} + \lambda_{\text{ref}}(1-2\epsilon) \geq w - 2\frac{w\sqrt{\lambda_{\text{ref}}}}{\sqrt{2}\lambda_{\text{ref}}^{1/4} w^{1/4}} + \lambda_{\text{ref}}(1-2\epsilon)$$

$$= w - \sqrt{2}w^{3/4}\lambda_{\text{ref}}^{1/4} + \lambda_{\text{ref}}(1-2\epsilon) =: f_\epsilon(w) =: f(w). \tag{2.4}$$

Then

$$f'(w) = 1 - \frac{3\lambda_{\text{ref}}^{1/4}}{2\sqrt{2}w^{1/4}},$$

and thus f is minimised at $w_* = 81\lambda_{\text{ref}}/64$ and

$$f(w_*) = \left(\frac{37}{64} - 2\epsilon\right)\lambda_{\text{ref}}.$$

For any $\lambda_{\text{ref}} > 0$ we can choose $\epsilon > 0$ small enough so that $f(w_*) > 0$. From (2.3) we can choose K large enough, so that for all $|x| > K$ and all v such that $\langle \nabla U(x), v\rangle > 0$

$$2\frac{\widetilde{\mathcal{L}}V(x,v)}{V(x,v)} < -\delta,$$

for some $\delta = f(w_*) > 0$.

Case $\langle \nabla U(x), v\rangle < 0$. Then from (2.2)

$$2\frac{\widetilde{\mathcal{L}}V(x,v)}{V(x,v)} = -\Bigg[\langle \nabla U(x), -v\rangle + 2\lambda_{\text{ref}} - \frac{1}{\lambda_{\text{ref}} + \langle \nabla U(x), -v\rangle}\langle v, \Delta U(x) v\rangle$$
$$- \lambda_{\text{ref}}\frac{\sqrt{\lambda_{\text{ref}} + \langle \nabla U(x), -v\rangle}}{\sqrt{\lambda_{\text{ref}}}}$$
$$- 2\lambda_{\text{ref}}\int_{\langle \nabla U(x), w\rangle \geq 0} \psi(\,\mathrm{d}w)\frac{\sqrt{\lambda_{\text{ref}} + \langle \nabla U(x), -v\rangle}}{\sqrt{\lambda_{\text{ref}} + \langle \nabla U(x), w\rangle}}\Bigg],$$

and arguing in the same way as in the previous case, given $\epsilon > 0$ we can choose $K > 0$ such that for all $|x| > K$ we have similarly to (2.3)

$$\int_{\langle \nabla U(x), w\rangle \geq 0} \psi(\,\mathrm{d}w) \frac{1}{\sqrt{\lambda_{\text{ref}} + \langle \nabla U(x), w\rangle}} \leq \int_{\langle \nabla U(x), w\rangle \geq 0} \psi(\,\mathrm{d}w) \frac{1}{\sqrt{\langle \nabla U(x), w\rangle}}$$
$$= \int_{\theta=0}^{\pi/2} \frac{p_\vartheta\,(\theta)\,\mathrm{d}\theta}{\sqrt{|\nabla U(x)|\cos(\theta)}} \leq \frac{\epsilon}{\lambda_{\text{ref}}}.$$

Since $\overline{\lim}_{|x|\to\infty} \|\Delta U(x)\| \leq \alpha_1$, for K large enough and $|x| > K$ we have $\|\Delta U(x)\| \leq 2\alpha_1$. Thus overall when $\langle \nabla U(x), v\rangle < 0$

$$\frac{\widetilde{\mathcal{L}}V(x,v)}{V(x,v)} \leq -\frac{1}{2}\Bigg[\langle \nabla U(x), -v\rangle + 2\lambda_{\text{ref}} - \frac{2}{\lambda_{\text{ref}} + \langle \nabla U(x), -v\rangle}\alpha_1$$
$$- \lambda_{\text{ref}} \frac{\sqrt{\lambda_{\text{ref}} + \langle \nabla U(x), -v\rangle}}{\sqrt{\lambda_{\text{ref}}}} - 2\epsilon\sqrt{\lambda_{\text{ref}} + \langle \nabla U(x), -v\rangle}\Bigg].$$

For $w = \langle \nabla U(x), -v\rangle > 0$ define

$$g(w) := w + 2\lambda_{\text{ref}} - \frac{2\alpha_1}{\lambda_{\text{ref}} + w} - \lambda_{\text{ref}} \frac{\sqrt{\lambda_{\text{ref}} + w}}{\sqrt{\lambda_{\text{ref}}}} - 2\epsilon\sqrt{\lambda_{\text{ref}} + w},$$
$$g'(w) = 1 + \frac{2\alpha_1}{(\lambda_{\text{ref}} + w)^2} - \frac{\sqrt{\lambda_{\text{ref}}}}{2\sqrt{\lambda_{\text{ref}} + w}} - \frac{\epsilon}{\sqrt{\lambda_{\text{ref}} + w}}$$
$$\geq 1 + \frac{2\alpha_1}{(w + \lambda_{\text{ref}})^2} - \frac{1}{2} - \frac{\epsilon}{\sqrt{\lambda_{\text{ref}}}} = \frac{1}{2} + \frac{2\alpha_1}{(w + \lambda_{\text{ref}})^2} - \frac{\epsilon}{\sqrt{\lambda_{\text{ref}}}},$$

and thus for all λ_{ref} we can choose ϵ small enough so that $g'(w) \geq 0$ for all $w \geq 0$. Therefore

$$g(w) \geq g(0) = \lambda_{\text{ref}} - 2\epsilon\sqrt{\lambda_{\text{ref}}} - \frac{2\alpha_1}{\lambda_{\text{ref}}}.$$

If $\lambda_{\text{ref}} \geq (2\alpha_1 + 1)^2$ then for ϵ small enough we have that $g(w) \geq \delta > 0$, for some δ.

Thus, there exists $K > 0$ large enough so that for all $|x| > K$ and v such that $\langle \nabla U(x), v\rangle < 0$ we have $\widetilde{\mathcal{L}}V(x,v)/V \leq -\delta$. Therefore ($\mathfrak{D}$) holds with $C = B(0,K) \times \mathbb{S}^{d-1}$.

*Condition (*B*).* Recall that $2\alpha_2 := \underline{\lim}_{|x|\to\infty} |\nabla U(x)|$, so that we can choose K large enough so that for all $|x| > K$ we have $|\nabla U(x)| \geq \alpha_2$. Thus when $|x| > K$

$$\int_{\langle \nabla U(x), w\rangle \geq 0} \frac{\psi(\,\mathrm{d}w)}{\sqrt{\lambda_{\text{ref}} + \langle \nabla U(x), w\rangle}} = \int_{\theta=0}^{\pi/2} \frac{p_\vartheta(\,\mathrm{d}\theta)}{\sqrt{\lambda_{\text{ref}} + |\nabla U(x)|\cos(\theta)}}$$

$$\leq \frac{1}{\sqrt{\lambda_{\mathrm{ref}}}} \int_{\theta=0}^{\pi/2} \frac{p_\vartheta(\,\mathrm{d}\theta)}{\sqrt{1+\frac{\alpha_2}{\lambda_{\mathrm{ref}}}\cos(\theta)}}$$
$$= \frac{1}{\sqrt{\lambda_{\mathrm{ref}}}} F\left(\frac{\alpha_2}{\lambda_{\mathrm{ref}}}, d\right),$$

where
$$F\left(u,d\right) := \mathbb{E}\left[\frac{\mathbb{I}\left(\vartheta \in [0,\pi/2]\right)}{\sqrt{1+u\cos\vartheta}}\right], \quad \vartheta \sim p_\vartheta\left(\cdot\right),$$

with p_ϑ as defined in equation (2.2) of the main manuscript. Clearly $F(u,d) \leq F(0,d) = 1/2$ for all u and $F(u,d) \to 0$ as $u \to \infty$ for all d. Thus we can choose a sequence $\{c_d\}_{d\geq 0}$ such that $F(c_d,d) \leq 1/4$. One such choice given in the statement of the Theorem is $c_d = 16\sqrt{d}$. Indeed as $d \to \infty$,

$$F(c_d,d) = \kappa_d \int_{\theta=0}^{\pi/2} \frac{(\sin\theta)^{d-2}\,\mathrm{d}\theta}{\sqrt{1+c_d\cos\theta}}$$
$$\leq \frac{1}{4d^{1/4}} \int_{\theta=0}^{\pi/2} \frac{\kappa_d(\sin\theta)^{d-2}\,\mathrm{d}\theta}{\sqrt{\cos\theta}}$$
$$\leq \frac{\kappa_d}{4d^{1/4}} \int_{\theta=0}^{\pi/2} (\sin\theta)^{d-2}(\cos\theta)^{-1/2}\,\mathrm{d}\theta = \frac{1}{8}\frac{1}{d^{1/4}}\frac{\mathrm{Beta}\left(\frac{d-1}{2},\frac{1}{4}\right)}{\mathrm{Beta}\left(\frac{d-1}{2},\frac{1}{2}\right)} < \frac{1}{4}.$$

Case $\langle\nabla U(x), v\rangle = 0$. For $|x| > K$ and (2.2) we have

$$2\frac{\widetilde{\mathcal{L}}V(x,v)}{V(x,v)} \leq -\left[\lambda_{\mathrm{ref}} - 2\lambda_{\mathrm{ref}} \int_{\theta=0}^{\pi/2} \frac{\sqrt{\lambda_{\mathrm{ref}}}p_\vartheta\left(\theta\right)\,\mathrm{d}\theta}{\sqrt{\lambda_{\mathrm{ref}} + |\nabla U(x)|\cos(\theta)}}\right]$$
$$\leq -\lambda_{\mathrm{ref}}\left(1 - 2F\left(\alpha_2/\lambda_{\mathrm{ref}}, d\right)\right)$$
$$\leq -\lambda_{\mathrm{ref}}\left(1 - 2\times\frac{1}{4}\right) = -\frac{\lambda_{\mathrm{ref}}}{2},$$

as long as $\alpha_2/\lambda_{\mathrm{ref}} > c_d$, with c_d defined as in the statement of Theorem 3.1(B).

Case $\langle\nabla U(x), v\rangle > 0$. From (2.2) we have

$$2\frac{\widetilde{\mathcal{L}}V(x,v)}{V(x,v)} \leq -\left[\langle\nabla U(x), v\rangle - 2\frac{\langle\nabla U(x), v\rangle\sqrt{\lambda_{\mathrm{ref}}}}{\sqrt{\lambda_{\mathrm{ref}} + \langle\nabla U(x), v\rangle}} + \lambda_{\mathrm{ref}} - 2\lambda_{\mathrm{ref}} F\left(\frac{\alpha_2}{\lambda_{\mathrm{ref}}}, d\right)\right].$$

For $w \geq 0$, using again that $\lambda_{\mathrm{ref}} + w \geq 2\sqrt{w}\sqrt{\lambda_{\mathrm{ref}}}$, we have

$$w - 2\frac{w\sqrt{\lambda_{\mathrm{ref}}}}{\sqrt{\lambda_{\mathrm{ref}} + w}} + \lambda_{\mathrm{ref}}(1-2\epsilon) \geq w - \sqrt{2}\lambda_{\mathrm{ref}}^{1/4}w^{3/4} + \lambda_{\mathrm{ref}}(1-2\epsilon) = f_\epsilon(w).$$

Recall from (2.4) that for all $w \geq 0$

$$f_\epsilon(w) \geq \lambda_{\text{ref}} \left(\frac{37}{64} - 2\epsilon \right) > 0,$$

as long as $\epsilon < 37/128$. For each d and $\alpha_2 > 0$ we can choose λ_{ref} small enough so that $F(\alpha_2/\lambda_{\text{ref}}, d) < 37/128$. Then following a similar reasoning as before it can be easily seen that, as long as λ_{ref} is small enough, then there exists a $\delta > 0$, such that for all $|x| > K$ and $\langle \nabla U(x), v \rangle > 0$ we have

$$2 \frac{\widetilde{\mathcal{L}} V(x,v)}{V(x,v)} \leq -\delta.$$

Case $\langle \nabla U(x), v \rangle < 0$. Suppose that $\lambda_{\text{ref}} \leq \alpha_2/c_d$ or equivalently that $\alpha_2/\lambda_{\text{ref}} \geq c_d$. Then since $\|\Delta U(x)\| \to 0$, for all $\epsilon_1 > 0$, there is a $K > 0$ such that for all $|x| > K$ and λ_{ref} small enough

$$\begin{aligned}
&2 \frac{\widetilde{\mathcal{L}} V(x,v)}{V(x,v)} \\
&\leq -\Bigg[\langle \nabla U(x), -v \rangle + 2\lambda_{\text{ref}} - \frac{1}{\lambda_{\text{ref}} + \langle \nabla U(x), -v \rangle} \langle v, \Delta U(x) v \rangle \\
&\qquad - \lambda_{\text{ref}} \frac{\sqrt{\lambda_{\text{ref}} + \langle \nabla U(x), -v \rangle}}{\sqrt{\lambda_{\text{ref}}}} - 2\lambda_{\text{ref}} \frac{\sqrt{\lambda_{\text{ref}} + \langle \nabla U(x), -v \rangle}}{\sqrt{\lambda_{\text{ref}}}} F(\alpha_2/\lambda_{\text{ref}}, d) \Bigg] \\
&\leq -\Bigg[\langle \nabla U(x), -v \rangle + 2\lambda_{\text{ref}} - \frac{1}{\lambda_{\text{ref}} + \langle \nabla U(x), -v \rangle} \langle v, \Delta U(x) v \rangle \\
&\qquad - \lambda_{\text{ref}} \frac{\sqrt{\lambda_{\text{ref}} + \langle \nabla U(x), -v \rangle}}{\sqrt{\lambda_{\text{ref}}}} - 2\lambda_{\text{ref}} \frac{\sqrt{\lambda_{\text{ref}} + \langle \nabla U(x), -v \rangle}}{\sqrt{\lambda_{\text{ref}}}} F(c_d, d) \Bigg] \\
&\leq -\Bigg[\langle \nabla U(x), -v \rangle + 2\lambda_{\text{ref}} - \frac{\epsilon_1}{\lambda_{\text{ref}} + \langle \nabla U(x), -v \rangle} \\
&\qquad - \lambda_{\text{ref}} \frac{\sqrt{\lambda_{\text{ref}} + \langle \nabla U(x), -v \rangle}}{\sqrt{\lambda_{\text{ref}}}} - \frac{1}{2} \sqrt{\lambda_{\text{ref}}} \sqrt{\lambda_{\text{ref}} + \langle \nabla U(x), -v \rangle} \Bigg].
\end{aligned}$$

Let $w = \langle \nabla U(x), -v \rangle > 0$ and consider

$$g(w) := w + 2\lambda_{\text{ref}} - \frac{\epsilon_1}{\lambda_{\text{ref}} + w} - \lambda_{\text{ref}} \frac{\sqrt{\lambda_{\text{ref}} + w}}{\sqrt{\lambda_{\text{ref}}}} - \frac{1}{2} \sqrt{\lambda_{\text{ref}}} \sqrt{\lambda_{\text{ref}} + w}.$$

Then we obtain

$$g'(w) = 1 + \frac{\epsilon_1}{(\lambda_{\text{ref}} + w)^2} - \frac{3\sqrt{\lambda_{\text{ref}}}}{4\sqrt{\lambda_{\text{ref}} + w}}. \geq 0.$$

Thus, we have

$$g(w) \geq g(0) = \frac{\lambda_{\text{ref}}}{2} - \frac{\epsilon_1}{\lambda_{\text{ref}}},$$

which is strictly positive as long as $\epsilon_1 < \lambda_{\text{ref}}^2/2$, and the result follows. □

PROOF OF LEMMA 5. First notice that V with $\lambda_{\text{ref}}(x)$ as defined in the statement of the Lemma also belongs to $\mathcal{D}(\widetilde{\mathcal{L}})$ from the same arguments as in the proof of Lemma 4. We now prove that V satisfies ($\mathfrak{D}$). From the form of ($\mathfrak{D}$) it follows that we can assume without loss of generality that $|x| > 1$, so that

$$\Lambda_{\text{ref}}(x) = \lambda_{\text{ref}} + \frac{|\nabla U(x)|}{|x|^{\epsilon}}.$$

First we restrict our attention to the case where $\langle \nabla U(x), v \rangle \neq 0$, for which we compute

$$\begin{aligned}
\frac{\partial V}{\partial x_i} &= \frac{1}{2} V(x,v) U_{x_i}(x) \\
&\quad - \frac{1}{2} \frac{\mathrm{e}^{U(x)/2}}{(\Lambda_{\text{ref}}(x) + \langle \nabla U(x), -v \rangle_+)^{3/2}} \left[\frac{\partial}{\partial x_i} \Lambda_{\text{ref}}(x) + \frac{\partial}{\partial x_i} \langle \nabla U(x), -v \rangle_+ \right], \\
\langle \nabla V, v \rangle &= \frac{1}{2} V(x,v) \langle \nabla U(x), v \rangle \\
&\quad - \frac{1}{2} \frac{\mathrm{e}^{U(x)/2}}{(\Lambda_{\text{ref}}(x) + \langle \nabla U(x), -v \rangle_+)^{3/2}} \left[\langle \nabla \Lambda_{\text{ref}}(x), v \rangle \right. \\
&\quad \left. - \langle v, \Delta U(x) v \rangle \mathbb{1}\{\langle \nabla U(x), -v \rangle > 0\} \right] \\
&= \frac{1}{2} V(x,v) \left\{ \langle \nabla U(x), v \rangle - \frac{\langle \nabla \Lambda_{\text{ref}}(x), v \rangle}{\Lambda_{\text{ref}}(x) + \langle \nabla U(x), -v \rangle_+} \right. \\
&\quad \left. + \frac{\langle v, \Delta U(x) v \rangle \mathbb{1}\{\langle \nabla U(x), -v \rangle > 0\}}{\Lambda_{\text{ref}}(x) + \langle \nabla U(x), -v \rangle_+} \right\}.
\end{aligned}$$

After adding the reflection and refreshment terms we get

$$\begin{aligned}
\widetilde{\mathcal{L}} V(x,v) &= \frac{1}{2} V(x,v) \left\{ \langle \nabla U(x), v \rangle - \frac{\langle \nabla \Lambda_{\text{ref}}(x), v \rangle}{\Lambda_{\text{ref}}(x) + \langle \nabla U(x), -v \rangle_+} \right. \\
&\quad \left. + \frac{\langle v, \Delta U(x) v \rangle \mathbb{1}\{\langle \nabla U(x), -v \rangle > 0\}}{\Lambda_{\text{ref}}(x) + \langle \nabla U(x), -v \rangle_+} \right\} \\
&\quad + \Lambda_{\text{ref}}(x) \int V(x,w) \psi(\mathrm{d}w) - \Lambda_{\text{ref}}(x) V(x,v) \\
&\quad + \langle \nabla U(x), v \rangle \mathbb{1}\{\langle \nabla U(x), v \rangle \geq 0\} \left[V(x, R(x)v) - V(x,v) \right],
\end{aligned}$$

and thus

$$\begin{aligned}\frac{\widetilde{\mathcal{L}}V(x,v)}{V(x,v)} &= \frac{1}{2}\bigg\{\langle \nabla U(x), v\rangle - \frac{\langle \nabla \Lambda_{\mathrm{ref}}(x), v\rangle}{\Lambda_{\mathrm{ref}}(x) + \langle \nabla U(x), -v\rangle_+} \\ &\quad + \frac{\langle v, \Delta U(x) v\rangle \mathbb{1}\{\langle \nabla U(x), -v\rangle > 0\}}{\Lambda_{\mathrm{ref}}(x) + \langle \nabla U(x), -v\rangle_+}\bigg\} \\ &\quad + \Lambda_{\mathrm{ref}}(x) \int \left[\frac{\sqrt{\Lambda_{\mathrm{ref}}(x) + \langle \nabla U(x), -v\rangle_+}}{\sqrt{\Lambda_{\mathrm{ref}}(x) + \langle \nabla U(x), w\rangle_+}} - 1\right] \psi(\,\mathrm{d}w) \\ &\quad + \langle \nabla U(x), v\rangle \mathbb{1}\{\langle \nabla U(x), v\rangle \geq 0\} \left[\frac{\sqrt{\Lambda_{\mathrm{ref}}(x) + \langle \nabla U(x), -v\rangle_+}}{\sqrt{\Lambda_{\mathrm{ref}}(x) + \langle \nabla U(x), v\rangle_+}} - 1\right].\end{aligned}$$

Thus when $\langle \nabla U(x), v\rangle > 0$ we have

$$\begin{aligned}\frac{\widetilde{\mathcal{L}}V(x,v)}{V(x,v)} &= \frac{1}{2}\langle \nabla U(x), v\rangle - \frac{1}{2}\frac{\langle \nabla \Lambda_{\mathrm{ref}}(x), v\rangle}{\Lambda_{\mathrm{ref}}(x)} \\ &\quad + \langle \nabla U(x), v\rangle \left[\frac{\sqrt{\Lambda_{\mathrm{ref}}(x)}}{\sqrt{\Lambda_{\mathrm{ref}}(x) + \langle \nabla U(x), v\rangle}} - 1\right] \\ &\quad + \Lambda_{\mathrm{ref}}(x) \int_{\langle \nabla U(x), w\rangle \geq 0} \left[\frac{\sqrt{\Lambda_{\mathrm{ref}}(x)}}{\sqrt{\Lambda_{\mathrm{ref}}(x) + \langle \nabla U(x), w\rangle_+}} - 1\right] \psi(\,\mathrm{d}w).\end{aligned} \tag{2.5, 2.6}$$

When $\langle \nabla U(x), v\rangle < 0$ then

$$\begin{aligned}\frac{\widetilde{\mathcal{L}}V(x,v)}{V(x,v)} &= \frac{1}{2}\bigg\{\langle \nabla U(x), v\rangle - \frac{\langle \nabla \Lambda_{\mathrm{ref}}(x), v\rangle - \langle v, \Delta U(x) v\rangle}{\Lambda_{\mathrm{ref}}(x) + \langle \nabla U(x), -v\rangle}\bigg\} \\ &\quad + \Lambda_{\mathrm{ref}}(x) \int \left[\frac{\sqrt{\Lambda_{\mathrm{ref}}(x) + \langle \nabla U(x), -v\rangle_+}}{\sqrt{\Lambda_{\mathrm{ref}}(x) + \langle \nabla U(x), w\rangle_+}} - 1\right] \psi(\,\mathrm{d}w).\end{aligned} \tag{2.7}$$

When $\langle \nabla U(x), v\rangle = 0$, similarly to the proof of Lemma 4, by considering separately the case where $\langle \Delta U(x), -v\rangle > 0$ and $\langle \Delta U(x), -v\rangle \leq 0$ we find that

$$\begin{aligned}&\lim_{t\to 0+} \frac{\mathrm{d}}{\mathrm{d}t} V(x+tv, v) \\ &= \lim_{t\to 0+} \frac{1}{t}\left\{V(x+tv, v) - V(x,v)\right\} \\ &= \lim_{t\to 0+} \frac{1}{t}\left\{\frac{\exp\left(U(x+tv)/2\right)}{\sqrt{\Lambda_{\mathrm{ref}}(x+tv) + \langle \nabla U(x+tv), -v\rangle_+}} - \frac{\exp\left(U(x)/2\right)}{\sqrt{\Lambda_{\mathrm{ref}}(x)}}\right\}\end{aligned}$$

$$= -\frac{1}{2}\frac{\exp\left(U(x)/2\right)}{\sqrt{\Lambda_{\mathrm{ref}}(x)}^3}\left[\langle \nabla \Lambda_{\mathrm{ref}}(x), v\rangle + \langle \Delta U(x) v, -v\rangle_+\right]$$
$$= -\frac{V(x,v)}{2}\frac{\left[\langle \nabla \Lambda_{\mathrm{ref}}(x), v\rangle + \langle \Delta U(x) v, -v\rangle_+\right]}{\Lambda_{\mathrm{ref}}(x)}.$$

Thus for $\langle \nabla U(x), v\rangle = 0$, after adding the refreshment term we have

$$\frac{\widetilde{\mathcal{L}}V(x,v)}{V(x,v)} = -\frac{1}{2}\frac{\langle \nabla \Lambda_{\mathrm{ref}}(x), v\rangle + \langle \Delta U(x) v, -v\rangle_+}{\Lambda_{\mathrm{ref}}(x)}$$
$$+ \Lambda_{\mathrm{ref}}(x)\int\left[\frac{\sqrt{\Lambda_{\mathrm{ref}}(x)}}{\sqrt{\Lambda_{\mathrm{ref}}(x) + \langle \nabla U(x), w\rangle_+}} - 1\right]\psi(\,\mathrm{d}w) \tag{2.8}$$
$$\leq -\frac{1}{2}\frac{\langle \nabla \Lambda_{\mathrm{ref}}(x), v\rangle}{\Lambda_{\mathrm{ref}}(x)} \tag{2.9}$$
$$+ \Lambda_{\mathrm{ref}}(x)\int\left[\frac{\sqrt{\Lambda_{\mathrm{ref}}(x)}}{\sqrt{\Lambda_{\mathrm{ref}}(x) + \langle \nabla U(x), w\rangle_+}} - 1\right]\psi(\,\mathrm{d}w).$$

From the definition of $\Lambda_{\mathrm{ref}}(x)$ and the chain rule

$$\nabla \Lambda_{\mathrm{ref}}(x) = |x|^{-\epsilon}\nabla|\nabla U(x)| + |\nabla U(x)|\nabla\left(|x|^{-\epsilon}\right).$$

We first compute

$$\frac{\partial}{\partial x_i}|\nabla U(x)| = \frac{\partial}{\partial x_i}\sqrt{\sum_j\left(\frac{\partial}{\partial x_j}U\right)^2} = |\nabla U(x)|^{-1}\sum_j \frac{\partial}{\partial x_j}U(x)\frac{\partial^2}{\partial x_i \partial x_j}U(x),$$

whence it follows that

$$|\nabla\,|\nabla U|| = |\nabla U|^{-1}|\left\{\sum_{i=1}^d\left[\sum_{j=1}^d \frac{\partial}{\partial x_j}U(x)\frac{\partial^2}{\partial x_i \partial x_j}U(x)\right]^2\right\}^{1/2}$$
$$\leq |\nabla U(x)|^{-1}|\nabla U|\|\Delta U\| = \|\Delta U\|$$

Thus we have that

$$\frac{|\nabla \Lambda_{\mathrm{ref}}(x)|}{|\Lambda_{\mathrm{ref}}(x)|} \leq \frac{\|\Delta U(x)\|/|x|^\epsilon}{|\nabla U(x)|/|x|^\epsilon} + \frac{|\nabla U(x)|}{|x|^{1+\epsilon}\times|\nabla U(x)|/|x|^\epsilon}$$
$$= \frac{\|\Delta U(x)\|}{|\nabla U(x)|} + \frac{1}{|x|} \to 0,$$

where we also used the fact that $|\nabla(|x|^{-\epsilon})| = \epsilon|x|^{-1-\epsilon}$. It therefore follows that

$$\overline{\lim_{|x|\to\infty}}\frac{|\langle \nabla \Lambda_{\mathrm{ref}}(x), v\rangle|}{\Lambda_{\mathrm{ref}}(x)} = 0, \tag{2.10}$$

so that this term can be ignored for large $|x|$. Also notice that

$$\begin{aligned}
&\int_{w:\langle\nabla U(x),w\rangle\geq 0}\psi(\,\mathrm{d}w)\frac{1}{\sqrt{\Lambda_{\mathrm{ref}}(x)+\langle\nabla U(x),w\rangle_+}}\\
&=\frac{1}{|\nabla U(x)|^{1/2}}\int_{w:\langle\nabla U(x),w\rangle\geq 0}\frac{\psi(\,\mathrm{d}w)}{\sqrt{\frac{\Lambda_{\mathrm{ref}}(x)}{|\nabla U(x)|}+\left\langle\frac{\nabla U(x)}{|\nabla U(x)|},w\right\rangle}}\\
&=\frac{1}{|\nabla U(x)|^{1/2}}\int_{\theta=0}^{\pi/2}\frac{p_\vartheta(\,\mathrm{d}\theta)}{\sqrt{\frac{\Lambda_{\mathrm{ref}}(x)}{|\nabla U(x)|}+\cos(\theta)}},
\end{aligned}$$

where d is the dimension. As $|x|\to\infty$, our definition of $\Lambda_{\mathrm{ref}}(x)$ ensures that

$$\int_{\theta=0}^{\pi/2}\frac{p_\vartheta(\,\mathrm{d}\theta)}{\sqrt{\frac{\Lambda_{\mathrm{ref}}(x)}{|\nabla U(x)|}+\cos(\theta)}}\to\int_{\theta=0}^{\pi/2}\frac{p_\vartheta(\,\mathrm{d}\theta)}{\sqrt{\cos(\theta)}}=-\frac{3\Gamma\left(-\frac{3}{4}\right)\Gamma\left(\frac{d}{2}\right)}{8\sqrt{\pi}\Gamma\left(\frac{d}{2}-\frac{1}{4}\right)}=:\gamma_d>0.$$

Case $\langle\nabla U(x),v\rangle=0$. Thus when $\langle\nabla U(x),v\rangle=0$, for $|x|$ large we have

$$\begin{aligned}
\frac{\widetilde{\mathcal{L}}V(x,v)}{V(x,v)}&\sim\Lambda_{\mathrm{ref}}(x)\int\left[\frac{\sqrt{\Lambda_{\mathrm{ref}}(x)}}{\sqrt{\Lambda_{\mathrm{ref}}(x)+\langle\nabla U(x),w\rangle_+}}-1\right]\psi(\,\mathrm{d}w)\\
&\sim\Lambda_{\mathrm{ref}}(x)\int_{\langle\nabla U(x),w\rangle<0}[1-1]\,\psi(\,\mathrm{d}w)+\Lambda_{\mathrm{ref}}(x)\left[\frac{\sqrt{\Lambda_{\mathrm{ref}}(x)}}{\sqrt{|\nabla U(x)|}}\gamma_d-\frac{1}{2}\right]\\
&\sim\Lambda_{\mathrm{ref}}(x)\left[\frac{\sqrt{\Lambda_{\mathrm{ref}}(x)}}{\sqrt{|\nabla U(x)|}}\gamma_d-\frac{1}{2}\right],
\end{aligned}$$

and from the definition of $\Lambda_{\mathrm{ref}}(x)$ it easily follows that

$$\frac{\widetilde{\mathcal{L}}V(x,v)}{V(x,v)}\sim-\frac{1}{2}\Lambda_{\mathrm{ref}}(x)\to-\infty.$$

Case $\langle\nabla U(x),v\rangle>0$. For $|x|$ large we have

$$\begin{aligned}
\frac{\widetilde{\mathcal{L}}V(x,v)}{V(x,v)}\sim\frac{1}{2}\langle\nabla U(x),v\rangle+\langle\nabla U(x),v\rangle\left[\frac{\sqrt{\Lambda_{\mathrm{ref}}(x)}}{\sqrt{\Lambda_{\mathrm{ref}}(x)+\langle\nabla U(x),v\rangle}}-1\right]\\
+\Lambda_{\mathrm{ref}}(x)\left[\frac{\sqrt{\Lambda_{\mathrm{ref}}(x)}}{\sqrt{|\nabla U(x)|}}\gamma_d-\frac{1}{2}\right].
\end{aligned}$$

Using the definition of $\Lambda_{\mathrm{ref}}(x)$, and letting $\langle\nabla U(x),v\rangle=|\nabla U(x)|\cos(\theta)$ for $\theta\in[0,\pi/2)$ we have for $\langle\nabla U(x),v\rangle>0$ as $|x|\to\infty$

$$\frac{\widetilde{\mathcal{L}}V(x,v)}{V(x,v)}\sim\frac{1}{2}|\nabla U(x)|\cos(\theta)$$

$$
\begin{aligned}
&+ |\nabla U(x)| \cos(\theta) \left[\frac{\sqrt{|\nabla U(x)|/|x|^\epsilon}}{\sqrt{|\nabla U(x)|/|x|^\epsilon + |\nabla U(x)|\cos(\theta)}} - 1 \right] \\
&+ \frac{|\nabla U(x)|}{|x|^\epsilon} \left[\frac{\sqrt{|\nabla U(x)|/|x|^\epsilon}}{\sqrt{|\nabla U(x)|}} \gamma_d - \frac{1}{2} \right] \\
= {}& |\nabla U(x)| \left[\cos(\theta) \left(\frac{1}{2} + \frac{1}{\sqrt{1 + |x|^\epsilon \cos(\theta)}} - 1 \right) + \frac{1}{|x|^\epsilon} \left[\frac{\gamma_d}{|x|^{\epsilon/2}} - \frac{1}{2} \right] \right] \\
= {}& \frac{|\nabla U(x)|}{|x|} |x| \left[\cos(\theta) \left(-\frac{1}{2} + \frac{1}{\sqrt{1 + |x|^\epsilon \cos(\theta)}} \right) + \frac{1}{|x|^\epsilon} \left[\frac{\gamma_d}{|x|^{\epsilon/2}} - \frac{1}{2} \right] \right] \\
\leq {}& |x| \left[\cos(\theta) \left(-\frac{1}{2} + \frac{1}{\sqrt{1 + |x|^\epsilon \cos(\theta)}} \right) + \frac{1}{|x|^\epsilon} \left[\frac{\gamma_d}{|x|^{\epsilon/2}} - \frac{1}{2} \right] \right],
\end{aligned}
$$

since $|\nabla U(x)|/|x| \to \infty$ and the quantity in brackets is clearly negative for large enough $|x|$. Let $u = \cos(\theta)$ and $r = |x|$. Then observe that we can rewrite the right hand side as

$$
\begin{aligned}
&r^{1-\epsilon} r^\epsilon u \left(\frac{1}{\sqrt{1 + r^\epsilon u}} - \frac{1}{2} \right) - \frac{r^{1-\epsilon}}{2} + O(r^{1/4}) \\
&\qquad \leq \frac{r^{1-\epsilon}}{4} - \frac{r^{1-\epsilon}}{2} + O(r^{1-3\epsilon/2}) = -\frac{\sqrt{r}}{4} + O(r^{1-3\epsilon/2}),
\end{aligned}
$$

since for $w = r^\epsilon u > 0$ it can be shown that

$$
w \left(\frac{1}{\sqrt{1+w}} - \frac{1}{2} \right) \leq \frac{1}{4}.
$$

Thus it follows that for $\langle \nabla U(x), v \rangle > 0$ we have that $\overline{\lim}_{|x| \to \infty} \widetilde{\mathcal{L}} V/V = -\infty$.

Case $\langle \nabla U(x), v \rangle < 0$. From (2.7) and (2.10) we have as $|x| \to \infty$

$$
\begin{aligned}
&2 \overline{\lim_{|x| \to \infty}} \frac{\widetilde{\mathcal{L}} V(x,v)}{V(x,v)} \\
&= \overline{\lim_{|x| \to \infty}} \Bigg\{ \langle \nabla U(x), v \rangle - \frac{\langle \nabla \Lambda_{\mathrm{ref}}(x), v \rangle - \langle v, \Delta U(x) v \rangle}{\Lambda_{\mathrm{ref}}(x) + \langle \nabla U(x), -v \rangle} \\
&\qquad + 2\Lambda_{\mathrm{ref}}(x) \int \left[\frac{\sqrt{\Lambda_{\mathrm{ref}}(x) + \langle \nabla U(x), -v \rangle_+}}{\sqrt{\Lambda_{\mathrm{ref}}(x) + \langle \nabla U(x), w \rangle_+}} - 1 \right] \psi(\,\mathrm{d}w) \Bigg\} \\
&= \overline{\lim_{|x| \to \infty}} \Bigg\{ \langle \nabla U(x), v \rangle + \frac{\langle v, \Delta U(x) v \rangle}{\Lambda_{\mathrm{ref}}(x) + \langle \nabla U(x), -v \rangle}
\end{aligned}
$$

$$
\begin{aligned}
&+ \Lambda_{\mathrm{ref}}(x)\left[\frac{\sqrt{\Lambda_{\mathrm{ref}}(x)+\langle\nabla U(x),-v\rangle}}{\sqrt{\Lambda_{\mathrm{ref}}(x)}}-1\right]\\
&+ 2\Lambda_{\mathrm{ref}}(x)\frac{\sqrt{\Lambda_{\mathrm{ref}}(x)+\langle\nabla U(x),-v\rangle_+}}{\sqrt{|\nabla U(x)|}}\gamma_d-\Lambda_{\mathrm{ref}}(x)\Bigg\}\\
= \overline{\lim_{|x|\to\infty}}&\Bigg\{\langle\nabla U(x),v\rangle+\Lambda_{\mathrm{ref}}(x)\left[\frac{\sqrt{\Lambda_{\mathrm{ref}}(x)+\langle\nabla U(x),-v\rangle}}{\sqrt{\Lambda_{\mathrm{ref}}(x)}}-1\right]\\
&+ 2\Lambda_{\mathrm{ref}}(x)\frac{\sqrt{\Lambda_{\mathrm{ref}}(x)+\langle\nabla U(x),-v\rangle_+}}{\sqrt{|\nabla U(x)|}}\gamma_d-\Lambda_{\mathrm{ref}}(x)\Bigg\},
\end{aligned}
$$

since $\overline{\lim}_{|x|\to\infty}\|\Delta U(x)\|/\Lambda_{\mathrm{ref}}(x)\to 0$. Thus letting θ be the angle between $U(x)$ and $-v$, we have

$$
\begin{aligned}
&2\overline{\lim_{|x|\to\infty}}\frac{\widetilde{\mathcal{L}}V(x,v)}{V(x,v)}\\
&=\overline{\lim_{|x|\to\infty}}\Bigg\{-|\nabla U(x)|\cos(\theta)+\frac{|\nabla U(x)|}{|x|^\epsilon}\left[\frac{\sqrt{|\nabla U(x)|/|x|^\epsilon+|\nabla U(x)|\cos(\theta)}}{\sqrt{|\nabla U(x)|/|x|^\epsilon}}-1\right]\\
&\qquad+2\frac{|\nabla U(x)|}{|x|^\epsilon}\left(\frac{\sqrt{|\nabla U(x)|/|x|^\epsilon+|\nabla U(x)|\cos(\theta)}}{\sqrt{|\nabla U(x)|}}\gamma_d-\frac{1}{2}\right)\Bigg\}\\
&=\overline{\lim_{|x|\to\infty}}\Bigg\{-|\nabla U(x)|\cos(\theta)+\frac{|\nabla U(x)|}{|x|^\epsilon}\left[\sqrt{1+|x|^\epsilon\cos(\theta)}-1\right]\\
&\qquad+2\frac{|\nabla U(x)|}{|x|^\epsilon}\left(\sqrt{\frac{1}{|x|^\epsilon}+\cos(\theta)}\gamma_d-\frac{1}{2}\right)\Bigg\}\\
&=\overline{\lim_{|x|\to\infty}}|\nabla U(x)|\Bigg\{-\cos(\theta)+\frac{1}{|x|^\epsilon}\left[\sqrt{1+|x|^\epsilon\cos(\theta)}-1\right]+\\
&\qquad\frac{2}{|x|^\epsilon}\left(\sqrt{\frac{1}{|x|^\epsilon}+\cos(\theta)}\gamma_d-\frac{1}{2}\right)\Bigg\}\\
&=\overline{\lim_{|x|\to\infty}}\frac{|\nabla U(x)|}{|x|}|x|\Bigg\{-\cos(\theta)+\frac{1}{|x|^\epsilon}\left[\sqrt{1+|x|^\epsilon\cos(\theta)}-1\right]\\
&\qquad+\frac{2}{|x|^\epsilon}\left(\sqrt{\frac{1}{|x|^\epsilon}+\cos(\theta)}\gamma_d-\frac{1}{2}\right)\Bigg\}\\
&\le\overline{\lim_{|x|\to\infty}}|x|\Bigg\{-\cos(\theta)+\frac{1}{|x|^\epsilon}\left[\sqrt{1+|x|^\epsilon\cos(\theta)}-1\right]
\end{aligned}
$$

$$+ \frac{2}{|x|^\epsilon} \left(\sqrt{\frac{1}{|x|^\epsilon} + \cos(\theta)}\gamma_d - \frac{1}{2} \right) \Bigg\},$$

since the right hand side is clearly negative for $|x|$ large enough.

For $u = \cos(\theta) \in [0, 1]$ define the function

$$f(u) := -u + \frac{1}{|x|^\epsilon} \left[\sqrt{1 + |x|^\epsilon u} - 1 \right] + \frac{2}{|x|^\epsilon} \left(\sqrt{\frac{1}{|x|^\epsilon} + u}\gamma_d - \frac{1}{2} \right).$$

Then

$$f'(u) = -1 + \frac{|x|^\epsilon}{2|x|^\epsilon \sqrt{1 + |x|^\epsilon u}} + \frac{1}{|x|^\epsilon \sqrt{\frac{1}{|x|^\epsilon} + u}} \gamma_d.$$

This is negative for all $u \geq 0$ for $|x|$ large enough. Therefore

$$f(u) \leq f(0) = \frac{2}{|x|^\epsilon} \left(\sqrt{\frac{1}{|x|^\epsilon}}\gamma_d - \frac{1}{2} \right) \sim -\frac{1}{|x|^\epsilon},$$

as $|x| \to \infty$. Hence

$$\varlimsup_{|x| \to \infty} \frac{\widetilde{\mathcal{L}} V(x, v)}{V(x, v)} = -\infty,$$

and the result follows. □

Proof of Lemma 6. *Checking Assumption* (A0'). Notice that from equations (3.6), (3.7) and (3.8) of the main manuscript, the functions $h^{(i)}$, are infinitely differentiable except perhaps for $x = 0$ and $|x| = 1/b$ for $i = 1$, or $|x| = R$ for $i = 2$. Thus U_h will satisfy Assumption (A0) for $|x|$ large enough and in fact everywhere except for $|x| = 0, 1/b$ for $i = 1$, and $|x| = 0, R$ for $i = 2$. It remains to show that the mapping $t \mapsto \langle \nabla U_h(x + tv), v \rangle$ is locally Lipschitz at these points. First, from the definition of $f = f^{(i)}$, it follows easily that the mapping $t \mapsto \langle \nabla U_h(x + tv), v \rangle$ will be continuous and piecewise smooth, and thus locally Lipschitz, at $|x| = 1/b$ and $|x| = R$ for $i = 1, 2$ respectively. To deal with the remaining case $x = 0$, we next show that $t \mapsto \langle \nabla U_h(tv), v \rangle$ is in fact differentiable at $t = 0$.

Recall the decomposition of ∇U_h given in (3.10) of the main manuscript. The first term of (3.10) is given by

$$\nabla \log \det(\nabla h(x)) = \begin{cases} \left[\frac{f''(|x|)}{f'(|x|)} + (d-1) \left(\frac{f'(|x|)}{f(|x|)} - \frac{1}{|x|} \right) \right] \frac{x}{|x|}, & x \neq 0, \\ 0, & x = 0, \end{cases} \tag{2.11}$$

whence we can compute

$$\frac{1}{t} \left[\langle \nabla \log \det(\nabla h(tv)), v \rangle - \langle \nabla \log \det(\nabla h(0)), v \rangle \right]$$

$$
\begin{aligned}
&= \frac{1}{t}\left[\frac{f''(t)}{f'(t)} + (d-1)\left(\frac{f'(t)}{f(t)} - \frac{1}{t}\right)\right]\left\langle \frac{tv}{|tv|}, v\right\rangle \\
&= \frac{1}{t}\left[\frac{f''(t)}{f'(t)} + (d-1)\left(\frac{f'(t)}{f(t)} - \frac{1}{t}\right)\right].
\end{aligned}
$$

In the case $f = f^{(1)}$ we have

$$
f(0) = 0, \quad f'(0) = \frac{b\mathrm{e}}{2}, \quad f''(0) = 0, \quad f'''(0) = b^3\mathrm{e},
$$

and thus using Taylor expansions

$$
\begin{aligned}
&\lim_{t\to 0}\frac{1}{t}\left[\frac{f''(t)}{f'(t)} + (d-1)\left(\frac{f'(t)}{f(t)} - \frac{1}{t}\right)\right] \\
&= 0 + (d-1)\lim_{t\to 0}\frac{1}{t^2 f(t)}[tf'(t) - f(t)] \\
&= (d-1)\lim_{t\to 0}\frac{1}{t^3 f'(0)}\Big(tf'(0) + t^2 f''(0) + \frac{t^3}{2}f'''(0) - f(0) \\
&\qquad - tf'(0) - \frac{t^2}{2}f''(0) - \frac{t^3}{6}f'''(0) + o(t^3)\Big) \\
&= (d-1)\lim_{t\to 0}\frac{1}{t^3 f'(0)}\left(\frac{t^3}{2}f'''(0) - \frac{t^3}{6}f'''(0) + o(t^3)\right) = \frac{(d-1)}{3}\frac{f'''(0)}{f'(0)}.
\end{aligned}
$$

In the case $f = f^{(2)}$, we have for $t > 0$ small enough

$$
\frac{1}{t}\left[\frac{f''(t)}{f'(t)} + (d-1)\left(\frac{f'(t)}{f(t)} - \frac{1}{t}\right)\right] = 0.
$$

Thus overall $t \mapsto \langle \nabla \log\det(\nabla h(tv)), v\rangle$ is differentiable at $t = 0$ and thus locally Lipschitz.

We now deal with the second term of (3.10). From (5.1) we have

$$
\begin{aligned}
&\frac{1}{t}\left[\langle \nabla h(tv)\nabla U(h(tv)), v\rangle - \langle \nabla h(0)\nabla U(0), v\rangle\right] \\
&= \frac{1}{t}\left[\langle \nabla h(tv)\nabla U(h(tv)), v\rangle - f'(0)\langle \nabla U(0), v\rangle\right] \\
&= \frac{1}{t}\left[\frac{f(t)}{t}\langle \nabla U(h(tv)), v\rangle - f'(0)\langle \nabla U(0), v\rangle\right] + \frac{1}{t}\left[f'(t) - \frac{f(t)}{t}\right]\frac{\langle tv, v\rangle\langle \nabla U(h(tv)), tv\rangle}{t^2} \\
&= I_1 + I_2.
\end{aligned}
$$

For the first term we have

$$
I_1 = \frac{1}{t}\frac{f(t) - tf'(0)}{t}\langle \nabla U(h(tv)), v\rangle + \frac{1}{t}f'(0)\left[\langle \nabla U(h(tv)), v\rangle - \nabla U(0), v\rangle\right].
$$

Since U satisfies (A0) and h is differentiable, the second term of I_1 clearly converges. The first term also converges since ∇U is continuous and

$$\frac{f(t) - tf'(0)}{t^2} = \frac{f(0) + tf'(0) + t^2 f''(0) + o(t^2) - tf'(0)}{t^2}.$$

For the second term we have

$$\begin{aligned}
I_2 &= \frac{1}{t}\left[f'(t) - \frac{f(t)}{t}\right]\langle \nabla U(h(tv)), v\rangle \\
&= \frac{f'(t)t - f(t)}{t^2}\langle \nabla U(h(tv)), v\rangle \\
&= \frac{f'(0)t + f''(0)t^2 + o(t^2) - f(0) - tf'(0) - \frac{t^2}{2}f''(0)}{t^2}\langle v, v\rangle\langle \nabla U(h(tv)), v\rangle \\
&\to 0.
\end{aligned}$$

It follows that $t \mapsto \langle \nabla h(x+tv)\nabla U(h(x+tv)), v\rangle$ is differentiable at $t = 0$.

Checking Assumption (A1). For both $h = h^{(1)}$ and $h = h^{(2)}$, a change of variable leads to

(2.12)
$$\begin{aligned}
&\int \pi_h(y)|\nabla U_h(y)|\,\mathrm{d}y \\
&= \int \pi_h(h^{-1}(x))|\nabla h^{-1}(x)||\nabla U_h(h^{-1}(x))|\,\mathrm{d}x \\
&= \int \pi(h(h^{-1}(x)))|\nabla U_h(h^{-1}(x))||\nabla h(h^{-1}(x))||\nabla h^{-1}(x)|\,\mathrm{d}x
\end{aligned}$$

(2.13)
$$\begin{aligned}
&= \int \pi(x)|\nabla U_h(h^{-1}(x))|\,\mathrm{d}x \\
&\le \int \pi(x)[|\nabla\{h\}(h^{-1}(x))\nabla U(x)| + |\nabla \log\det(\nabla\{h\}(h^{-1}(x)))|]\,\mathrm{d}x
\end{aligned}$$

(2.14)
$$\le \int \pi(x)[\|\nabla\{h\}(h^{-1}(x))\||\nabla U(x)| + |\nabla \log\det(\nabla\{h\}(h^{-1}(x)))|]\,\mathrm{d}x.$$

Here for clarity we use the notation $\nabla\{\cdot\}(x)$ for the gradient of the function in the bracket evaluated at x and we will similarly use $\Delta\{\cdot\}(x)$ for its Hessian. We begin with the first term in (2.14). Under the assumptions of Theorem 3.3(A) we have, for $|x| > R$ and some constant $C > 0$, that $|\nabla U(x)| \le C|x|^{-1}$ and thus

$$\int \pi(x)|\nabla U(x)|\|\nabla\{h\}(h^{-1}(x))\|\,\mathrm{d}x \le C + C\int_{|x|>R} \pi(x)\frac{1}{|x|}f(|h^{-1}(x)|)\,\mathrm{d}x$$

$$\leq C + C\int_{|x|>R} \pi(x)\frac{1}{|x|}|x|\,\mathrm{d}x \leq 2C,$$

since clearly $f(|h^{-1}(x)|) = |x|$.

Under the assumptions of Theorem 3.3(B), by Assumption (B)-(ii), we can assume that there exists $K > 0$ such that if $|x| > K$ then $\langle x, \nabla U(x)\rangle \geq C|x|^\beta$ for some $C > 0$. Thus for $|x|$ large enough, say $K/|x| < 1/2$, we have

$$\begin{aligned} U(x) &= U\left(K\frac{x}{|x|}\right) + \int_{t=K/|x|}^{1} \frac{\mathrm{d}U(tx)}{\mathrm{d}t}\,\mathrm{d}t \\ &\geq U\left(K\frac{x}{|x|}\right) + \int_{t=1/2}^{1} \frac{1}{t}\langle \nabla U(tx), tx\rangle\,\mathrm{d}t \\ &\geq U\left(K\frac{x}{|x|}\right) + C\int_{t=1/2}^{1} \frac{1}{t}t^\beta |x|^\beta\,\mathrm{d}t \geq C|x|^\beta, \end{aligned} \tag{2.15}$$

since $U \geq 0$. Therefore

$$\begin{aligned} \int \pi(x)|\nabla U(x)|\|\nabla\{h\}(h^{-1}(x))\|\,\mathrm{d}x &\leq C\Big[1 + \int_{|x|>R} \pi(x)|x|^{\beta-1} f(|h^{-1}(x)|)\,\mathrm{d}x\Big] \\ &\leq C\Big[1 + \int_{|x|>R} \mathrm{e}^{-C|x|^\beta}|x|^\beta\,\mathrm{d}x\Big] < \infty. \end{aligned}$$

For the second term of (2.14), let

$$L'(x) := |\nabla \log\det(\nabla h(x))|\,.$$

From equation (5.2) of the main manuscript it follows easily that L' is bounded for both $h = h^{(1)}$ and $h = h^{(2)}$, and thus

$$\int \pi(x)|\nabla \log\det(\nabla\{h\}(h^{-1}(x)))|\,\mathrm{d}x < \infty.$$

Checking Assumption (A2). For $h = h^{(1)}$, notice that by [3, Lemma 4], and the fact that $h(\cdot)$ is isotropic in the sense of [3], it follows that

$$\overline{\lim_{|y|\to\infty}}\,|\nabla \log\det(\nabla h(y))| < C,$$

for some $C > 0$. Therefore

$$\begin{aligned} |\nabla U_h(y)| &\leq |\nabla h(y)\nabla U(h(y))| + |\nabla \log\det(\nabla h(y))| \\ &\leq \|\nabla h(y)\|\,|\nabla U(h(y))| + C \end{aligned}$$

and using Assumption (A)-(i) and equation (5.1) of the main manuscript

$$\leq C\,\|\nabla h(y)\|\,/|h(y)| + C \leq C,$$

since $\|\nabla h(y)\| \le C|h(y)|$. Thus it follows that

$$\frac{\mathrm{e}^{U_h(y)/2}}{\sqrt{|\nabla U_h(y)|}} \ge C\mathrm{e}^{U_h(y)/2} \to \infty,$$

as $|y| \to \infty$ since $\mathrm{e}^{-U_h(y)}$ is integrable.

On the other for $h = h^{(2)}$ notice that by [3, Lemma 2], and the fact that $h(\cdot)$ is isotropic in the sense of [3], we obtain

$$\overline{\lim_{|y|\to\infty}} |\nabla \log \det(\nabla h(y))| = 0. \tag{2.16}$$

Therefore

$$\begin{aligned}|\nabla U_h(y)| &\le |\nabla h(y) \nabla U(h(y))| + |\nabla \log \det(\nabla h(y))| \\ &\le \|\nabla h(y)\| \, |\nabla U(h(y))| + C.\end{aligned}$$

From equation (5.1) of the main manuscript it follows that $\|\nabla h(y)\| \le C|y|^{p-1}$. Therefore, using Assumption (B)-(i)

$$\begin{aligned}|\nabla U_h(y)| &\le C \, \|\nabla h(y)\| \, |h(y)|^{\beta-1} + C \\ &\le C|y|^{p-1+p\beta-p} = C|y|^{p\beta-1}.\end{aligned}$$

Thus

$$\frac{\mathrm{e}^{U_h(y)/2}}{\sqrt{|\nabla U_h(y)|}} \ge C\frac{\mathrm{e}^{U(h(y))/2}}{\det(\nabla h(y))\sqrt{|y|^{p\beta-1}}}.$$

Finally, recalling (2.15), for $|x|$ large enough, say $K/|x| < 1/2$, we have $U(x) \ge C|x|^\beta$. Since by definition $h(y) \sim |y|^p$, and from equation (5.2) of the main manuscript $\det(\nabla h(y))$ grows at most polynomially, we obtain

$$\frac{\mathrm{e}^{U(h(y))/2}}{\det(\nabla h(y))\sqrt{|y|^{p\beta-1}}} \ge \frac{\mathrm{e}^{C|h(y)|^\beta/2}}{\det(\nabla h(y))\sqrt{|y|^{p\beta-1}}} \ge \frac{\mathrm{e}^{C|y|^{p\beta}/2}}{\det(\nabla h(y))\sqrt{|y|^{p\beta-1}}} \to \infty.$$

□

Proof of Lemma 7(A). For notational simplicity, we assume $b = 1$ but the argument can be generalized to other values. We start by establishing the first condition of Theorem 3.1(B), i.e. that U_h satisfies our definition of exponential tail behaviour. In the remaining, assume $|x| > b^{-1} = 1$.

By Assumption (A)-(i) and Cauchy-Schwartz, we have for $|x|$ large enough

$$\left|\frac{\langle x, \nabla U(x)\rangle}{|x|}\right| \le |\nabla U(x)| \le \frac{c_1}{|x|},$$

hence π is a *sub-exponentially light density* as defined in [3, p. 3052]. This combined with Assumption (A)-(iii), which is equivalent to [3, Eq. (17)], means that we can apply [3, Theorem 3] to obtain that π_h is an *exponentially light density* as defined in [3, p. 3052]. More specifically, from the proof of [3, Theorem 3] it follows that

$$\overline{\lim_{|x|\to\infty}} \frac{-\langle x, \nabla U_h(x)\rangle}{|x|} = -b(\mathfrak{d} - d) < 0.$$

Applying Cauchy-Schwartz again, we obtain

$$0 < b(\mathfrak{d} - d) = \underline{\lim}_{|x|\to\infty} \frac{\langle x, \nabla U_h(x)\rangle}{|x|} \le \underline{\lim}_{|x|\to\infty} |\nabla U_h(x)|,$$

which establishes the first condition of Theorem 3.1(B) with $\alpha_2 = b(\mathfrak{d}-d)/2$.

We now turn our attention to the Hessian condition of Theorem 3.1(B). We first decompose the norm of the Hessian as follows:

$$\|\Delta U_h(x)\| \le \|\Delta\{U \circ h\}(x)\| + \|\Delta\{\log\det(\nabla h(x))\}(x)\|. \tag{2.17}$$

From [3, Lemma 1], we have for $|x| \ge 1$

$$\log\det(\nabla h(x)) = |x| + (d-1)[\log(e^{|x|} - e/3) - \log(|x|)] =: L(|x|),$$

hence

$$\Delta\{\log\det(\nabla h(x))\}(x) = \frac{L''(|x|)}{|x|^2} x x^T + \frac{L'(|x|)}{|x|} I_d - \frac{L'(|x|)}{|x|^3} x x^T.$$

We have

$$L'(r) = 1 + (d-1)\left[\frac{e^r}{e^r - e/3} - \frac{1}{r}\right],$$
$$L''(r) = (1-d)\left[\frac{e^{r+1}/3}{(e^r - e/3)^2} + \frac{1}{r^2}\right],$$

so $|L'(r)| \to d$, $|L''(r)| \to 0$ and therefore

$$\overline{\lim_{|x|\to\infty}} \|\Delta U_h(x)\| \le \overline{\lim_{|x|\to\infty}} \|\Delta\{U \circ h\}(x)\| + \overline{\lim_{|x|\to\infty}} \|\Delta\{\log\det(\nabla h(x))\}(x)\|$$
$$= \overline{\lim_{|x|\to\infty}} \|\Delta\{U \circ h\}(x)\|.$$

To control this remaining term, we bound the operator norm with the Frobenius norm and write

$$\begin{aligned}\overline{\lim_{|x|\to\infty}} \|\Delta\{U\circ h\}(x)\|^2 &\le \overline{\lim_{|x|\to\infty}} \sum_{i=1}^d \sum_{j=1}^d |\partial_i\partial_j\{U\circ h\}(x)|^2 \\ &= \overline{\lim_{|x|\to\infty}} \sum_{i=1}^d \sum_{j=1}^d \left|\partial_i \left\{ \sum_{k=1}^d \partial_k\{U\}(h(x))\partial_j\{h_k\}(x) \right\}\right|^2,\end{aligned}$$

where we write $\partial_i\{\cdot\}(x)$ as a shorthand for the i-th partial derivative, $(\nabla\{\cdot\}(x))_i$.

It is enough to bound the d^2 expressions of the form

$$\begin{aligned}\left|\partial_i \left\{ \sum_{k=1}^d \partial_k\{U\}(h(x))\partial_j\{h_k\}(x) \right\}\right| \le \sum_{k=1}^d \Big[&|\partial_i\{\partial_k\{U\}\circ h\}(x)\}\partial_j\{h_k\}(x)| \\ &+ |\partial_k\{U\}(h(x))\partial_i\{\partial_j\{h_k\}(x)\}|\}\Big]. \end{aligned} \tag{2.18}$$

The first term in Equation (2.18) is controlled as follows:

$$|\partial_i\{\partial_k\{U\}\circ h\}(x)\partial_j\{h_k\}(x)| \tag{2.19}$$

$$\le |\partial_j\{h_k\}(x)| \sum_{m=1}^d |\partial_m\partial_k\{U\}(h(x))||\partial_i\{h_m\}(x)|. \tag{2.20}$$

Using again [3, Lemma 1], and the fact that $|h(x)| = f(|x|) \le f'(|x|)$, for $|x|$ large enough,

$$|\partial_i\{h_j\}(x)| = \left| \frac{f(|x|)}{|x|}\mathbf{1}[i=j] + \left[f'(|x|) - \frac{f(|x|)}{|x|}\right]\frac{x_i x_j}{|x|^2}\right| \le 3f(|x|), \tag{2.21}$$

hence using Assumption (A)-(ii), for $|x|$ large enough,

$$|\partial_i\{\partial_k\{U\}\circ h\}(x)\partial_j\{h_k\}(x)| \le d\frac{c_2}{(h(|x|))^2}(3f(|x|))^2.$$

The second term in Equation (2.18) is controlled similarly, this time using Assumption (A)-(i), for $|x|$ large enough,

$$|\partial_k\{U\}(h(x))\partial_i\{\partial_j\{h_k\}(x)\}| \le \frac{c_1}{h(|x|)}(8f(|x|)), \tag{2.22}$$

since it follows from [3, Lemma 1] that

$$|\partial_i\partial_j\{h_k\}(x)| \le 8f(|x|). \qquad \square$$

PROOF OF LEMMA 7(B). Let $f := f^{(2)}$, $h := h^{(2)}$ given in (3.7) and (3.8) of the main manuscript respectively. We need to check that the assumptions of Theorem 3.2 are satisfied. First we check that

$$\lim_{|x|\to\infty} \frac{|\nabla U_h(x)|}{|x|} = \infty.$$

From (2.16) and (3.10) of the main manuscript it follows that

$$\lim_{|x|\to\infty} \frac{|\nabla U_h(x)|}{|x|} = \lim_{|x|\to\infty} \frac{|\nabla h(x) \nabla U\left(h(x)\right)|}{|x|}.$$

Recall from [3, Lemma 1] that for $x \neq 0$

$$\nabla h(x) = \frac{f(|x|)}{|x|} \mathbb{1}_d + \left[f'(|x|) - \frac{f(|x|)}{|x|} \right] \frac{x x^T}{|x|^2}, \tag{2.23}$$

where $\mathbb{1}_d$ is the $d \times d$-identity matrix. Therefore we have

$$\begin{aligned} \nabla h(x) \nabla U\left(h(x)\right) &= \frac{f(|x|)}{|x|} \nabla U\left(h(x)\right) + \left[f'(|x|) - \frac{f(|x|)}{|x|} \right] \left\langle \nabla U\left(h(x)\right), \frac{x}{|x|} \right\rangle \frac{x}{|x|} \\ &= f'(|x|) \left\langle \nabla U\left(h(x)\right), \frac{x}{|x|} \right\rangle \frac{x}{|x|} + \frac{f(|x|)}{|x|} P_x^{\perp} \nabla U\left(h(x)\right), \end{aligned}$$

where $P_x^{\perp}$ denotes the orthogonal projection on the plane normal to x. Therefore, since by definition $h(x) := f(|x|)x/|x|$, we have that

$$\begin{aligned} |\nabla h(x) \nabla U\left(h(x)\right)| &\geq f'(|x|) \left| \left\langle \nabla U\left(h(x)\right), \frac{x}{|x|} \right\rangle \right| \\ &= \frac{f'(|x|)}{f(|x|)} \left| \left\langle \nabla U\left(h(x)\right), h(x) \right\rangle \right| \\ &= \frac{f'(|x|)}{f(|x|)} |h(x)|^{\beta} \left[|h(x)|^{-\beta} \left| \left\langle \nabla U\left(h(x)\right), h(x) \right\rangle \right| \right]. \end{aligned}$$

Since $|h(x)| \to \infty$ as $|x| \to \infty$, Assumption (B)-(ii) and the definitions of f and h yield

$$\begin{aligned} \lim_{|x|\to\infty} \frac{|\nabla U_h(x)|}{|x|} &\geq \lim_{|x|\to\infty} |x|^{-1} \left\{ \frac{f'(|x|)}{f(|x|)} |h(x)|^{\beta} \left[|h(x)|^{-\beta} \left| \left\langle \nabla U\left(h(x)\right), h(x) \right\rangle \right| \right] \right\} \\ &\geq C \lim_{|x|\to\infty} |x|^{-1-1+\beta p} = C|x|^{\beta p - 2} = \infty, \end{aligned} \tag{2.24}$$

since $\beta p > 2$.

Finally we need to check, that for some $\epsilon > 0$ we have

$$\lim_{|x|\to\infty} \frac{\|\Delta U_h(x)\|}{|\nabla U_h(x)|}|x|^\epsilon = 0.$$

Recall the expression (2.17). It follows easily from the definitions of h, f and [3, Lemma 1, Eq.(13)] that

$$\lim_{|x|\to\infty} \|\Delta \log\det(\nabla h)(x)\| = 0.$$

Therefore we focus on the first term of (2.17). As in the proof of the first part of the Theorem, we need essentially to control terms of the form (2.19) and terms of the form (2.22). To this end, using Assumption (B)-(iii), we estimate

$$\begin{aligned}
&|\partial_i\{\partial_k\{U\}\circ h\}(x)\partial_j\{h_k\}(x)| \\
&\leq |\partial_j\{h_k\}(x)| \sum_{m=1}^{d} |\partial_m\partial_k\{U\}(h(x))||\partial_i\{h_m\}(x)| \\
&\leq C|x|^{2p-2}|h(x)|^{\beta-2} \leq C|x|^{2p-2+p\beta-2p} = C|x|^{p\beta-2},
\end{aligned}$$

since from equation (5.1) of the main manuscript and the definitions of f and h one can easily show that $|\partial_i h_k(x)| \leq |x|^{p-1}$. On the other hand, from Assumption (B)-(i) and the fact that $|\partial_i\partial_j\{h_k\}(x)| \leq C|x|^{p-2}$, which follows again from (5.1), the remaining terms can be estimated through

$$|\partial_k\{U\}(h(x))\partial_i\{\partial_j\{h_k\}\}(x)| \leq |h(x)|^{\beta-1}|x|^{p-2} \leq C|x|^{p\beta-2}.$$

Therefore combining the above with the arguments leading to (2.24) we have that as $|x| \to \infty$

$$\frac{\|\Delta U_h(x)\|}{|\nabla U_h(x)|}|x|^\epsilon \leq C\frac{|x|^{\beta p-2}}{|x|^{\beta p-1}}|x|^\epsilon \to 0, \qquad \square$$

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