An Approximation Algorithm for Continuous LMPs

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The goal of this presentation: examine the main topics of LMP theory and discuss their algorithmic contents. In particular, a randomized algorithm that approximates continuous LMPs is described.

1 Representation of (Some) Continuous LMPs

In order to represent most kernels that occur in practical situations, we introduce the following assumption on our LMPs $(S, \mathfrak{M}, \boldsymbol{\tau})$:

Assumption 1. There is a "canonical" measure μ on $(S, \mathfrak{M})$ from which points can be sampled.

In this situation, we can represent kernels using probability density functions:

Definition 1. *A family of sub-probability density functions* $f_a : S^2 \to [0, \infty)$, $s \in S, a \in A$, which is simply a family of $(\mathfrak{M} \otimes \mathfrak{M})$-measurable functions such that $\int_S f_a(s_0, \cdot) d\mu \leq 1 \; \forall s_0 \in S, a \in A$. Then, the kernels are given by:

$$\tau_a(s_0, M) := \int_M f_a(s_0, \cdot) d\mu \; \forall M \in \mathfrak{M}, a \in A, s_0 \in S.$$

We will assume throughout this lecture that the kernels are of that form.

Numerical integration cannot be applied here because M could be "too nasty" geometrically to allow a nice partitioning. The solution comes from probability theory:

Theorem 1. *Let* $(\Omega, \mathscr{F}, P)$, $(S, \mathfrak{M}, \mu)$ *be probability spaces. Assume that we can sample the random variables* $X_1, X_2, \ldots, X_i : \Omega \to S$, *identically and independently according to the distribution* μ. *Then, if* $f : S \to \mathbb{R}$ *is integrable and* $M \in \mathfrak{M}$ *we have:*

$$\frac{1}{n} \sum_{i=1}^{n} (\chi_M \cdot f) \circ X_i \to \int_M f d\mu \quad (a.s.).$$

2 Computing the Approximation

The main difficulties encountered in the implementation of the rational tree approximation algorithm are:

1. given a measurable function f, compute the inverse image of an interval,
2. compute the infimum of the value attained by f on $M \in \mathfrak{M}$,
3. given $M_1, M_2 \in \mathfrak{M}$, determine whether their intersection is non-empty (denoted is-$\emptyset$; this is needed when a partition is generated from a set of partitions).

Since we use Monte Carlo integration for the representation of the kernels, the only operation we require on measurable sets is to test membership of a given point (in particular, there is no need for an operation that would express a measurable set as the union of intervals), so the first point is no longer problematic. The second point is treated by replacing infimum by *essential infimum*:

Definition 2. Let $(X, \mathfrak{G}, \mu)$ be a measure space. We define the *essential infimum* over $M \in \mathfrak{G}$ of a measurable function $f : (X, \mathfrak{G}) \to (\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ to be:

$$\operatorname*{ess\,inf}_{M} f := \sup \Big\{ a \in \mathbb{R} : \mu\Big(\big\{x \in M : f(x) < a\big\}\Big) = 0 \Big\}.$$

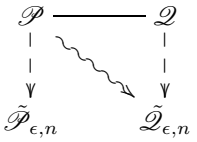
This has the advantage of being computable by a randomized algorithm. Similarly, the third point is treated by replacing the set-theoretic operation by the (computable) measure-theoretic analogue.

We next show this approximation of an approximation has desirable properties:

Theorem 2. *For all $\epsilon > 0, \epsilon \in \mathbb{Q}$, $n \in \mathbb{N}$, the* Monte Carlo rational approximation $\tilde{\mathscr{Q}}_{\epsilon,n}$ *(i.e., the approximation with* inf*'s replaced by ess* inf*'s and is-$\emptyset$ replaced by is-null) is computable and has the following properties:*

1. *every state (X, l) of $\tilde{\mathscr{Q}}_{\epsilon,n}$ is simulated in $\mathscr{P}$ by every $s \in X$,*
2. *if a state $s \in S$ satisfies a formula $\phi \in \mathscr{L}_{\bigvee}$, then there is some Monte Carlo approximation $\tilde{\mathscr{Q}}_{\epsilon_0,n_0}$ such that $(X_s, n) \models \phi$*

Proof.



is the object we want to construct, where the dashed lines denote approximation by the non Monte Carlo method, curved lines, approximation by the Monte Carlo method, and plain lines, bisimulation.

We first construct, for a given $\epsilon > 0$ and $n \in \mathbb{N}$, a new LMP $\mathscr{Q} = (S \cap Z^c, \sigma(S \cap Z^c), \boldsymbol{\tau}\big|_{S\cap Z^c \times \sigma(S\cap Z^c)})$ which differs only on a set Z of μ-measure zero. We define this set Z as follows:

$$Z := \bigcup_{\mathscr{S}_{\epsilon,n}} \Big(\bigcup_{(X,l+1)\in P,(B,l)\in P} \big\{x \in X : \tau_a(x, B) < \operatorname*{ess\,inf}_{t\in X} \tau_a(t, B)\big\}\Big),$$

where $\mathscr{S}_{\epsilon,n} = (P, P^2, \rho)$ ranges over all (non Monte Carlo) rational tree approximations of $\mathscr{P}$. It is easy to see that Z is a countable union of μ-null sets, and hence is indeed itself μ-null. Informally, Z covers all the sets in S that cause a disagreement between the Monte Carlo approximation and the (standard) approximation.

Observe, moreover, that by the way $\mathscr{Q}$ is constructed, we have: $\tilde{\mathscr{Q}}_{\epsilon,n}$ = (Monte Carlo approximation of $\mathscr{P}$) = ((non Monte Carlo) approximation of $\mathscr{Q}$), where "=" stands here for equality of the probability transition matrices.

The next step is to show that $\mathscr{P}$ and $\mathscr{Q}$ are bisimilar, or equivalently, that for each $\phi \in \mathscr{L}_0$ and $s \in S \cap Z^c$, $s \models_{\mathscr{P}} \phi$ iff its copy in $\mathscr{Q}$ also satisfies ϕ (in $\mathscr{Q}$). The proof is by induction on the structure of formulas. The cases $\phi = T$ and $\phi = \psi_1 \cap \psi_2$, $\psi_1, \psi_2 \in \mathscr{L}_0$ are trivial, so suppose $\phi =< a >_q \psi$, with $q > 0$ and $\psi \in \mathscr{L}_0$. If $s \in S \cap Z^c$ satisfies ϕ in $\mathscr{Q}$, then by the fact the state space of $\mathscr{Q}$ is included in the state space of $\mathscr{P}$, we have that the copy of s in the state space of $\mathscr{P}$ also satisfies ϕ. Conversely, suppose $s \in S$ satisfies ϕ in $\mathscr{P}$. Let $[[\psi]]_{\mathscr{P}}$ denotes the set of states in $\mathscr{P}$ that satisfy ψ. In particular, $\tau_a(s, [[\psi]]_{\mathscr{P}}) > q$. We thence have:

$$\tau_a(s, [[\psi]]_{\mathscr{Q}}) = \tau_a(s, [[\psi]]_{\mathscr{P}} \cap Z^c) = \tau_a(s, [[\psi]]_{\mathscr{P}}) - \tau_a(s, [[\psi]]_{\mathscr{P}} \cap Z) = \tau_a(s, [[\psi]]_{\mathscr{P}}) > q,$$

using the fact that $\boldsymbol{\tau} \ll \mu$.

Now that all the edges of the diagram are established, the theorem follows directly from the characterization theorems of bisimulation and simulation, together with the established fact that (non Monte Carlo) rational tree approximations satisfy property 1 and 2 . □