STATISTICS 200, Section 102

YET MORE PROBABILITY

September 28, 2009 (Week #4, Monday)

Multiplication Rules

Recall our definition of conditional probability:

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

Rearrange as:

$P(A \text{ and } B) = P(A)P(B|A)$

A general way to compute probability of two events happening together ('intersection').

Three or more events together?

Multiplication rules, continued

$$P(A \text{ and } B \text{ and } C) =$$

Multiplication rules, continued

$$
\begin{aligned}
P(A \text{ and } B \text{ and } C) &= P(A \text{ and } B)P(C|A \text{ and } B) \\
&=
\end{aligned}
$$

Multiplication rules, continued

$$
\begin{aligned}
P(A \text{ and } B \text{ and } C) &= P(A \text{ and } B)P(C|A \text{ and } B) \\
&= P(A)P(B|A)P(C|A \text{ and } B)
\end{aligned}
$$

Multiplication rules, examples

Seven tickets in hat - 3 winners, 4 losers. Will select three at random (no replacement!).

$P(\text{all three are winners}) =$

Subtle distinction

What if replaced pulled ticket before selecting next one? Or the number of tickets in the hat is enormous (but still 3/7-ths of them are winners), so that pulling one out has a negligible impact on the proportion of winners amongst those remaining.

Then

$P(\text{all three are winners}) =$

Also a more general rule for ‘or’

For **any** two events A and B,

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Intuition - Venn diagram

‘Proof’ - can be deduced from our existing rules

Conditional probability and the law

Let the event S be occurence of incriminating evidence.

For instance,
$S = \{\text{two apparent SIDS deaths in same family}\}$.
$S = \{\text{sells shares moments before company announces bad news}\}$.

Let I be the event that the person in question is innocent.

Judge and jury might be swayed if $P(S|I)$ is really small. For instance, there was a (flawed) claim in the Sally Clark case that $P(S|I) = 1$ in 72 million.

Which way around for conditioning?

Consider using $P(S|I)$ as a starting point to determining $P(I|S)$, which is arguably the relevant quantity - evidence for innocence/guilt.

$$P(I|S) = \frac{P(S \text{ and } I)}{P(S)}$$

Which way around for conditioning?

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$$
\begin{aligned}
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&= \frac{P(S \text{ and } I)}{P(S \text{ and } I) + P(S \text{ and } I^C)}
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&= \frac{P(S|I)P(I)}{P(S|I)P(I) + P(S|I^C)\{1 - P(I)\}}
\end{aligned}
$$

Have developed an 'evidence processor'

Inputs

- $P(S|I)$ - chance that an innocent person triggers the suspicious event
- $P(S|I^C)$ - chance that a guilty person triggers the suspicious event - equal one?
- $P(I)$ - chance that a randomly selected person is innocent - close to one?

Outputs

- $P(I|S)$ - describes strength of evidence for/against accused!

Evidence processor, continued

Say, for example, $P(S|I^C) = 1$. Then:

$P(S\|I)$	$P(I)$	$P(I\|S)$
10^{-8}	$1-10^{-7}$	0.09
10^{-8}	$1-10^{-6}$	0.01
10^{-7}	$1-10^{-6}$	0.09
10^{-6}	$1-10^{-6}$	0.50
10^{-5}	$1-10^{-5}$	0.50

Need $P(S|I)$ several orders of magnitude smaller than $P(I^C)$ - which is the proportion guilty in the population - in order to be convinced of guilt by seeing S occur.

Prosecutor's fallacy: interpreting small $P(S|I)$ as evidence of guilt, without considering $P(I)$.

Final thought

Going from conditioning one way around to the other, for instance, from $P(S|I)$ to $P(I|S)$, or more generally from **data given parameters** to **parameters given data**, is known as **Bayes theorem**.

Very general idea, and ongoing research area!