THE EXPECTATION-MAXIMIZATION (EM) ALGORITHM

Common theme: turn a harder fitting problem into an iterative sequence of easlier problems.

Fit GLM via iterative sequence of weighted linear regression fits.

Fit nonlinear regression model via iterative sequence of linear regression fits.

Fit model to actual data via iterative sequence of fits to enhanced (completed) data.

Basic Framework

Let D be the observable data.

Want to maximize log-likelihood: $\log f(D|\theta)$.

Let C be the complete data.

What does this mean? There is actually a probability model for $(C|\theta)$ [which induces the model for $(D|\theta)$]. But we don't get to see C, only observe $D = g(C)$, where $g()$ is 'many-to-one'.

Very commonly, $g(C_1, C_2) = C_1$ (i.e, C_2 is 'missing').

Maybe one is trying to fit a model for Y given X_1 and X_2, but for some subjects the X_1 value is 'missing.'

Also very commonly, $\log f(C|\theta)$ easy to maximize, $\log f(D|\theta)$ hard. Take advantage of this structure?

Basic EM algorithm

Iterative sequence of estimates $\theta^{(1)}, \theta^{(2)}, \ldots$

$$\theta^{(k+1)} \quad = \quad \mathrm{argmax}_\theta \;\; h(\theta; \theta^{(k)})$$

where

$$h(\theta; \theta^*) \quad = \quad E_{\theta^*} \left\{ \log f(C|\theta) | D \right\}.$$

Here the averaging (expectation) is with respect to the conditional distribution of the complete data given the observed data, assuming θ^* is the parameter value.

Note: one iteration involves an E-step (evaluate expectation) followed by an M-step (maximize).

Intuition?

What can be proved about this algorithm?

Example: Normal Mixture

$X_1, \ldots, X_n$ *iid* from a *mixture* of $N(\mu_1, \sigma_1^2)$ (weight p) and $N(\mu_2, \sigma_2^2)$ (weight $1 - p$).

So the density of a single X is

$$p\phi(x|\mu_1, \sigma_1) + (1 - p)\phi(x|\mu_2, \sigma_2).$$

Digression: Two distinct rationales/scenarios for using mixture models. Flexibility of distributional form versus belief in latent structure.

Have $\theta = (\mu_1, \sigma_1, \mu_2, \sigma_2, p)$. No closed form ML estimator (log of sum complicates things).

What would make data ‘complete’?

Knowledge of which components generated which observations!

Have $C = (X_i, J_i)_{i=1}^n$, but $D = (X_i)_{i=1}^n$.

Model for $C = (X, J)$ given θ?

· $J \in \{1, 2\}$ (probs p and $1 - p$)

· $X|J \sim N(\mu_J, \sigma_J^2)$

Clear that complete data model for $(X, J|\theta)$ implies the desired model for observable data $(X|\theta)$.

E-Step

Boils down to computing

$$\begin{aligned} w_i &= E_{\theta^{(k)}}\left(I\{J_i = 1\} | X_i = x_i\right) \\ &= Pr_{\theta^{(k)}}(J_i = 1 | X_i = x_i) \\ &= \frac{p^{(k)}\phi(x_i|\mu_1^{(k)}, \sigma_1^{(k)})}{p^{(k)}\phi(x_i|\mu_1^{(k)}, \sigma_1^{(k)}) + (1 - p^{(k)})\phi(x_i|\mu_2^{(k)}, \sigma_2^{(k)})} \end{aligned}$$

the ‘weight’ or ‘responsibility’ of component 1 for the i-th datapoint, assuming $\theta = \theta^{(k)}$.

Just Bayes theorem!

Then have a closed-form M-step. Standard (weighted) normal-linear ‘stuff’. For instance

$$\begin{aligned}
p^{(k+1)} &\leftarrow n^{-1}\sum_i w_i \\
\mu_1^{(k+1)} &\leftarrow \frac{\sum_i w_i x_i}{\sum_i w_i} \\
\mu_2^{(k+1)} &\leftarrow \frac{\sum_i (1-w_i) x_i}{\sum_i (1-w_i)} \\
\sigma_1^{(k+1)} &\leftarrow \left\{\frac{\sum_i w_i (x_i-\mu_1^{(k+1)})^2}{\sum_i w_i}\right\}^{1/2} \\
\sigma_2^{(k+1)} &\leftarrow \left\{\frac{\sum_i (1-w_i)(x_i-\mu_2^{(k+1)})^2}{\sum_i (1-w_i)}\right\}^{1/2}
\end{aligned}$$

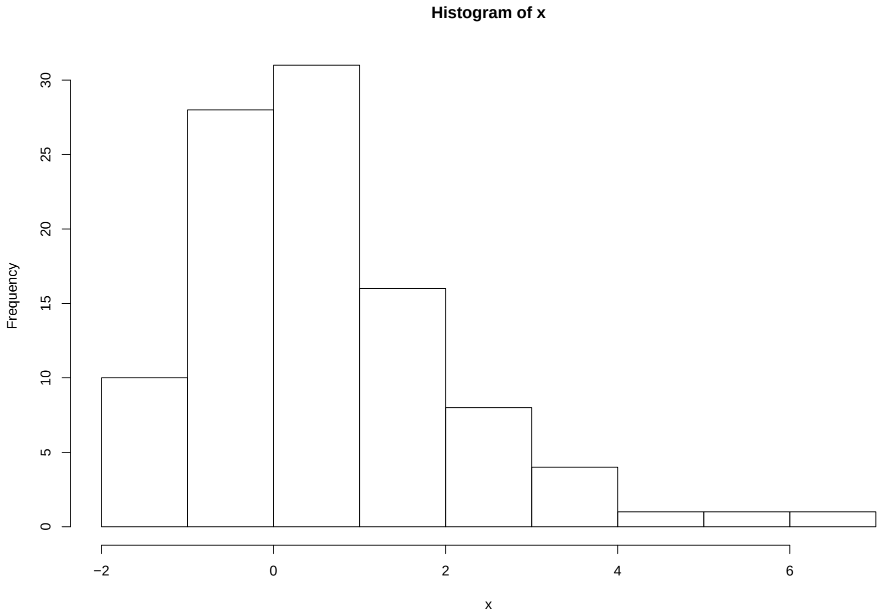
Easy to code!

```
normmix_function(x, numits=50) {
### starting vals
mu1 <- mean(x)-sqrt(var(x)); mu2 <- mean(x)+sqrt(var(x))
sig1 <- sig2 <- sqrt(var(x)/2); p <- 0.5

for (mnlp in 1:numits) {
  ### E-step
  wht <- p*dnorm(x, mu1, sig1) /
      (p*dnorm(x, mu1, sig1)+(1-p)*dnorm(x, mu2, sig2))
  ### M-step
  mu1 <- sum(wht*x)/sum(wht)
  mu2 <- sum((1-wht)*x)/sum(1-wht)
  sig1 <- sqrt( sum(wht*((x-mu1)^2))/sum(wht) )
  sig2 <- sqrt( sum((1-wht)*((x-mu2)^2))/sum((1-wht)) )
  p <- mean(wht) } }
```

Try it out!

```
> x <- c( rnorm(75), rnorm(25, 2, 2) )
> hist(x)
```



```
First 10 iterations

mu1      sig1     mu2    sig2       p loglik
-0.87    1.05    2.10    1.05    0.50 -197.07
-0.30    0.73    1.75    1.38    0.55 -173.88
-0.23    0.74    1.67    1.49    0.56 -172.83
-0.18    0.76    1.63    1.54    0.56 -172.40
-0.14    0.78    1.62    1.58    0.57 -172.17
-0.11    0.79    1.62    1.61    0.58 -172.03
-0.09    0.80    1.62    1.62    0.59 -171.93
-0.08    0.81    1.64    1.64    0.60 -171.86
-0.06    0.82    1.66    1.65    0.61 -171.79
-0.05    0.83    1.67    1.65    0.62 -171.74
```

Iterations 41-50, and 100

```
0.05    0.89    2.06    1.68    0.72 -171.39
0.05    0.89    2.06    1.68    0.72 -171.39
0.05    0.89    2.07    1.68    0.72 -171.39
0.05    0.89    2.07    1.68    0.72 -171.39
0.06    0.90    2.07    1.68    0.72 -171.39
0.06    0.90    2.08    1.68    0.72 -171.39
0.06    0.90    2.08    1.68    0.73 -171.39
0.06    0.90    2.08    1.68    0.73 -171.38
0.06    0.90    2.09    1.68    0.73 -171.38
0.06    0.90    2.09    1.68    0.73 -171.38

0.07    0.90    2.13    1.68    0.74 -171.38
```

What Else to Discuss?

What about standard errors?

Why does this work?

What other applications does it have?

What if the expectation in the E-step does not have a closed-form?