STAT 460/560- Assignments

Week I:

1. Let $X_1, X_2, \ldots, X_n$ be a random sample (iid) from a continuous distribution $f(x)$. Namely, the distribution family $\mathcal{F}$ contains all univariate continuous distributions.

   Let $R_1, R_2, \ldots, R_n$ be the rank statistic. That is, $R_1 =$ the rank of X_1 among n random variables.

   (a) Show that R is an ancillary statistic and find its distribution.

   (b) What information contained in R that might be useful for statistical inference?

2. Let $X_1, X_2, \ldots, X_n$ be a random sample (iid) from $N(\theta, \sigma^2)$. Let $\bar{X}_n$ and s_n^2 be the sample mean and variance.

   (a) Verify that $(X_1 - \bar{X}_n, \ldots, X_n - \bar{X}_n)/s_n$ is an ancillary statistic.

   (b) **Verify by factorization theorem that $\bar{X}_n, s_n^2$ are jointly sufficient**.

   (c) Suppose $\sigma = 1$ is known. Show that $\bar{X}_n$ is complete for θ by definition.

3. Let χ_d^2 and $\chi_d^2(\gamma^2)$ be two random variables with respectively central and non-central chi-square distributions with the same degrees of freedom d. Show that for any x,

$$P\{\chi_d^2(\gamma^2) \geq x^2\} \geq P\{\chi_d^2 \geq x^2\}.$$

   Suppose χ_n^2 is a third random variable with central chisquare distribution and n degrees of freedom. Then

$$F_{d,n}(\gamma) = \frac{\chi_d^2(\gamma^2)/d}{\chi_n^2/n}$$

   is said to have non-central F-distribution, when the two chisquare random variables are independent.

Show that for any x,

$$P\{F_{d,n}(\gamma^2) \geq x^2\} \geq P\{F_{d,n}(0) \geq x^2\}.$$

4. Let $\mathbf{Z}$ be a multivariate normal $N(0, \mathbb{I}_d)$ random vector. Show that for a symmetric matrix $\mathbf{A}$,

$$\mathbf{Z}^\tau \mathbf{A} \mathbf{Z}$$

has central chisquare distribution if and only if $\mathbf{A}^2 = \mathbf{A}$.

5. Let $\mathbf{Z}$ be a multivariate normal $N(0, \mathbb{I}_d)$ random vector. Under symmetry assumptions on $\mathbf{A}_1, \ldots, \mathbf{A}_p$, show that if

$$\mathbf{Z}^\tau \mathbf{Z} = \mathbf{Z}^\tau \mathbf{A}_1 \mathbf{Z} + \cdots + \mathbf{Z}^\tau \mathbf{A}_p \mathbf{Z} = \sum_{j=1}^{p} \mathbf{Q}_j$$

such that

$$\text{rank}(\mathbf{A}_1) + \cdots + \text{rank}(\mathbf{A}_p) = d$$

then $\mathbf{Q}_j$'s are independent, each has chisquare distribution of degrees equaling $\text{rank}(\mathbf{A}_j)$.