STAT 460/560- Assignments

Week III:

1. Let $X_1, X_2, ..., X_n$ be an i.i.d. sample from the Uniform distribution Unif$(0, \theta)$. Define $\hat{\theta}_n = \max\{X_1, X_2, \dots, X_n\}$, which is often denoted as $X_{(n)}$ and called order statistic.

   Find the limiting distribution of $n(\theta - \hat{\theta}_n)$ as $n \to \infty$.

   Is $\hat{\theta}$ asymptotically unbiased at rate $\sqrt{n}$, at rate n?

2. Let $X_1, X_2, ..., X_n$ be an i.i.d. random sample from Poisson (θ), and let $\eta = \exp(-\theta)$. From the previous assignment, we find that the UMVUE for η is given by

$$\hat{\eta} = (1 - 1/n)^{n\bar{X}}.$$

   (a) Follow the Definition 4.9 as given in the Lecture Notes, prove that $\hat{\eta}$ is weakly consistent, i.e., prove that, for any $\varepsilon > 0$ and $\theta > 0$,

$$P(|\hat{\eta} - \eta| > \varepsilon; \eta) \to 0,$$

   as $n \to \infty$.

   (Hint: Example 4.5 given in the class is a good starting point.)

   (b) Conduct a simulation study to find the probability in part (a). Let $\epsilon = 0.01, \eta = \exp(-1)$ and repeat the simulation with sample sizes $n = 100$ and 1000, with $N = 20000$ repetitions. Report your findings.

   (You can use the `R` code provided on Professor Chen's website, and play with the parameter value or ε value as you like.)

3. Let $X_1, X_2, ..., X_n$ be an i.i.d. sample from the following mixture model, with density function

$$f(x; \lambda, \pi) = (1 - \pi)\exp(-x) + \pi\lambda^{-1}\exp(-x/\lambda), \quad x > 0.$$

   Suppose we observe the sample data

```
0.61683384, 0.49301343, 0.08751571, 6.32112518, 1.46224603,
0.17420356, 1.07460011, 0.18795447, 2.01524287, 0.83013365,
0.04476622, 2.01365679, 1.63824658, 0.01627277, 5.71925356,
3.85095169, 0.75024996, 1.26231923, 0.70529060, 1.66594757
```

(a) Derive an analytical expression for moment estimate of the parameters λ and π and

(b) obtain their numerical values.

4. Given a positive constant k, let us define a function for the purpose of M-estimation:

$$\varphi(x;\theta) = \begin{cases} (x-\theta)^2 & , \text{ if } |x-\theta| \le k; \\ k^2 & , \text{ if } |x-\theta| > k. \end{cases}$$

(a) The M-Estimator $\hat{\theta}$ of θ is the value at which $M_n(\theta) = \sum_{i=1}^{n} \varphi(X_i, \theta)$ is minimized.

Assume that none of i makes $|X_i - \hat{\theta}| = k$ where $\hat{\theta}$ is the solution to the optimization problem.

Show that $\hat{\theta}$ is the mean of X_i such that $|X_i - \hat{\theta}| < k$.

(b) Given the sample data

```
1.551 -1.170 -0.201 1.143 0.138 3.103 1.455 -2.121 -1.672 6.150
```

and that $k = 2.0$, calculate the value of the M-Estimate defined in part (a).

5. Let $X_1, X_2, ..., X_n$ be an i.i.d. random sample from the Exponential distribution Exp (θ) with mean θ (the density is $f(x;\theta) = \theta^{-1}\exp(-x/\theta)$).

Denote by $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ the corresponding order statistics for this random sample. Then, $W_k = X_{(k+1)} - X_{(k)}, 1 \le k \le n-1$ are called the *spacings* of the order statistics. By convention, define $W_0 = X_{(1)}$, the first order statistic.

(a) It is known that $W_0, W_1, ..., W_{n-1}$ are independent to each other, with

$$W_k \sim Exp(\frac{\theta}{n-k}),$$

for $k = 0, 1, \ldots, n-1$.

Verify this Theorem for the case when $n = 2$.

(b) Let $T_n = X_{(1)} + X_{(2)} + \cdots + X_{(k)} + (n-k)X_{(k)}$.

Suppose $n = 10, k = 8$, find the mean and variance for this statistic T_n.

(Hint: You can use the Theorem stated in part (a).)

(c) Suppose $n = 10, k = 8$, and use on your result from part (b), give an unbiased L-Estimator for the parameter θ.

(You can refer to the Lecture Notes for the Definition of the L-estimator.)