STAT 460/560- Assignments

Week VII:

1. (a) Using R-package to plot 6 beta density functions with degrees of freedoms:

   $$(a, b) = (0.5, 0.1), (0.1, 0.5), (0.5, 0.5), (1, 1), (5, 1), (1, 5), (5, 20), (5, 50).$$

   (b) Select two additional pairs based on your own curiosity and plot them.

   (c) Show that the density functions with parameters (a, b) and (b, a) are mirror images of each other.

   Remark: (c) is an observational question.

2. Show that the following two pairs of distribution families permit analytical descriptions of the posterior distribution. Identify the posterior distribution together with specific parameter values.

   (a) Statistical model: Poisson with parameter θ; Prior distribution family: one parameter Gamma with degree of freedom d.

   (b) Statistical model: $\mathrm{N}(\mu, \sigma^2)$; Prior distribution family: $N(\mu_0, \sigma^2)$ for μ given σ^2, and one parameter Gamma for $1/\sigma^2$ with degree of freedom $d_0 = 5$.

   Remark: previous specification in (b) does not lead to easy description of the posterior distribution. Hence the previous claim of "conjugate" was false. Hint: work on conditional distribution of $1/\sigma^2$ given data and μ. The marginal distribution of μ is t.

3. Given a set of i.i.d. observations of size n from $\mathrm{N}(\mu, \sigma^2)$ and the prior distribution specified as in the previous problem with $\mu_0 = 0$.

   (a) Find the posterior 75% quantile of the mean parameter μ;

   (b) Find the posterior expectation of μ^2.

4. Following the last problem. Assume the data set contains $n = 20$ observations as follows:

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1.1777518 -0.5867896  0.2283789 -0.1735369 -0.2328192
1.0955114  1.2053680 -0.7216797 -0.3387580   0.1620835
1.4173256  0.0240219 -0.6647623  0.6214567   0.7466441
1.9525066 -1.2017093  1.9736293  -0.1168171  0.4511754
```

(a) Given $d_0 = 5$, plot the posterior mean of μ as a function of μ_0 over $[-2, 2]$.

(b) Given $\mu_0 = 0$, plot the posterior mean of σ^2 as a function of d_0 over $[0.5, 10]$.

Remark: use Monte Carlo simulation if direct numerical/analytical computation is too difficult/infeasible.

5. Based on the class example where the statistical model is Binomial and the prior distribution of θ is Beta(a, b).

(a) Compute the expected posterior squared loss of the MLE $\hat{\theta} = X/n$ and the Bayes estimator $\check{\theta} = (X + 1)/(n + 2)$. Remark: they are functions of (a, b).

(b) Compute the frequentist MSE of these two estimators. That is, regard the MSEs as functions of θ and θ non-random unknown parameter.