STAT 460/560- Assignments

Week VIII (and this is the last one for this term):

1. (a) Let Y be a random variable whose distribution is continuous with c.d.f. given by $F(y)$. Show that $U = F(Y)$ has uniform distribution on $[0, 1]$.

   (b) Find a function $g(\cdot)$ such that $g(U)$ has the standard Cauchy distribution when U has uniform distribution on $[0, 1]$.

2. Suppose we want to generate random numbers from standard normal distribution whose density is given by $\phi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$.

   We decide to use rejective sampling plan via double exponential distribution whose density function is given by:

   $$f_0(x) = \frac{1}{2} \exp(-|x|).$$

   (a) Compute the constant U as defined in our notes.

   (b) Write a code in R to implement the rejective sampling method to generate $n = 1000$ observations from N(0, 1).

   (c) Work out the Q-Q plot of the data generated and report the number of pairs of (X, U) in rejective sampling required.

   (d) How many pairs of (X, U) do you expect to be needed to generate $n = 1000$ normally distributed random numbers with this method?

3. In the Metropolis sampling algorithm, one choice of $r(\cdot, \cdot)$ function is

   $$r(x, y) = \min\big\{1, \frac{f(y)K_0(y, x)}{f(x)K_0(x, y)}\big\}.$$

   Show that this choice also leads to $f(x)$ satisfying the balance equation:

   $$f(x)K(x, y) = f(y)K(y, x).$$

4. Recall that we had difficulties to "identify" the marginal posterior distributions of μ, σ^2 when their priors are independent, one is normal and the other is inverse Gamma. Gibbs sampler may be a way to avoid the need of identifying it.

   (a) Use the data given in the last assignment, and prior $N(0, 4)$ for μ, Gamma($d_0 = 5$) for $\lambda = 1/\sigma^2$, write down the joint posterior density function of μ and σ^2 up to a multiplication constant.

   (b) Write a code to generate data by Gibbs sampler method from the above posterior distribution. Generate $N = 1000$ of pairs. Obtain their means.

   (c) Plot the density function in (a) and a density estimator based on posterior sample obtained in (b).

   Remark: make use of existing functions for generating Gamma, normal random numbers, and for density estimation.

   Feel free to share codes via Piazza.