Solution to Problem 8.16

8.16. a) For the double exponential probability density function

\[ f(x|\sigma) = \frac{1}{2\sigma} \exp \left( -\frac{|x|}{\sigma} \right), \]

the first population moment, the expected value of \( X \), is given by

\[ E(X) = \int_{-\infty}^{\infty} x \frac{1}{2\sigma} \exp \left( -\frac{|x|}{\sigma} \right) dx = 0 \]

because the integrand is an odd function \((g(-x) = -g(x))\). The first population moment does not depend on the unknown parameter \( \sigma \), so it cannot be used to develop a method of moments estimator of \( \sigma \). We go on to consider the second population moment:

\[ E(X^2) = \int_{-\infty}^{\infty} x^2 \frac{1}{2\sigma} \exp \left( -\frac{|x|}{\sigma} \right) dx = 2 \int_{0}^{\infty} x^2 \frac{1}{2\sigma} \exp \left( -\frac{x}{\sigma} \right) dx. \]

Re-expressing the last integral as the integral of a gamma density function leads to

\[ E(X^2) = 2\sigma^2 \]

as \( \int_{0}^{\infty} \left( \frac{1}{\sigma} \right)^3 x^{3-1} \exp \left( -\frac{x}{\sigma} \right) dx = 1 \). Solving for \( \sigma \) in terms of \( \mu_2 = E(X^2) \) leads to

\[ \sigma = \sqrt{\frac{1}{2} \mu_2}. \]

Therefore, the method of moments estimator of \( \sigma \) is given by

\[ \hat{\sigma}_{MM} = \sqrt{\frac{1}{2n} \sum_{i=1}^{n} X_i^2}. \]

b) The likelihood function is given by

\[ L(\sigma) = \prod_{i=1}^{n} f(x_i|\sigma) = \prod_{i=1}^{n} \frac{1}{2\sigma} \exp \left( -\frac{|x_i|}{\sigma} \right). \]

To simplify, consider the log-likelihood function

\[ l(\sigma) = \sum_{i=1}^{n} \left[ -\log(2\sigma) - \frac{|x_i|}{\sigma} \right] \]

\[ = -n \log 2 - n \log \sigma - \frac{\sigma}{\sigma} \sum_{i=1}^{n} |x_i| \]

\[ \Rightarrow l'(\sigma) = \frac{n}{\sigma} - \frac{1}{\sigma^2} \sum_{i=1}^{n} |x_i|. \]
It follows that
\[ l'(\sigma) > 0 \iff \sigma < \frac{1}{n} \sum_{i=1}^{n} |x_i| \]
so it is clear that the (unique) solution to \( l'(\sigma) = 0 \) corresponds to a maximum and we obtain
\[ \hat{\sigma}_{ML} = \frac{1}{n} \sum_{i=1}^{n} |X_i|. \]
c) If we let \( Y_i = X_i^2 \), we can write
\[ \hat{\sigma}_{MM} = \sqrt{\frac{1}{2n} \sum_{i=1}^{n} X_i^2} = \sqrt{\bar{Y}/2} = g(\bar{Y}), \]
where \( g(y) = \sqrt{y/2} \). The second order approximation to the expected value of \( \hat{\sigma}_{MM} \) is then is given by
\[ E(\hat{\sigma}_{MM}) \approx g(\mu_{\bar{Y}}) + \frac{1}{2} g''(\mu_{\bar{Y}}) \text{Var}(\bar{Y}) \]
But \( Y_1, \ldots, Y_n \) are independent and identically distributed (because \( X_1, \ldots, X_n \) are), so \( \mu_{\bar{Y}} = E(\bar{Y}) = E(Y) \) and \( \text{Var}(\bar{Y}) = \text{Var}(Y)/n \), leading to the simplified expression
\[ E(\hat{\sigma}_{MM}) \approx g(\mu_Y) + \frac{1}{2n} g''(\mu_Y) \text{Var}(Y). \]
From a), we obtain
\[ \mu_Y = E(Y) = E(X^2) = 2\sigma^2. \]
Also
\[ \text{Var}(Y) = \text{Var}(X^2) = E(X^4) - [E(X^2)]^2, \]
and
\[ E(X^4) = \int_{-\infty}^{\infty} x^4 \frac{1}{2\pi} \exp \left( -\frac{|x|}{\sigma} \right) dx = \frac{2}{2\sigma} \int_{0}^{\infty} x^4 e^{-\frac{x}{\sigma}} dx = 24\sigma^4, \]
after evaluating the integral by re-expressing it as a gamma density function. So
\[ \text{Var}(Y) = 24\sigma^4 - (2\sigma^2)^2 = 20\sigma^4, \]
and the expression becomes
\[ E(\hat{\sigma}_{MM}) \approx \sigma + \frac{10\sigma^4}{n} g''(2\sigma^2). \]
Finally, we evaluate the derivatives
\[ g'(y) = \frac{1}{2\sqrt{2}} y^{-\frac{1}{2}} \quad \text{and} \quad g''(y) = -\frac{1}{4\sqrt{2}} y^{-\frac{3}{2}}, \]
to obtain
\[ E(\hat{\sigma}_{MM}) \approx \sigma - \frac{5\sigma}{8n}. \]
and
\[ \text{Bias}(\hat{\sigma}_{MM}) \approx -\frac{5\sigma}{8n}. \]

As is typical for reasonable estimators, the bias is of order $1/n$. So, for large values of $n$, the MSE of $\hat{\sigma}_{MM}$ will be dominated by the variance (because $\text{MSE} = \text{Variance} + \text{Bias}^2$).

d) Because $X_1, \ldots, X_n$ are independent and identically distributed,
\[ E(\hat{\sigma}_{ML}) = E\left(\frac{1}{n} \sum_{i=1}^{n} |X_i|\right) = E(|X|). \]

But
\[ E(|X|) = \int_{-\infty}^{\infty} \frac{|x|}{2\sigma} \exp\left(-\frac{|x|}{\sigma}\right) dx = 2 \int_{0}^{\infty} \frac{x}{2\sigma} \exp\left(-\frac{x}{\sigma}\right) dx = \sigma, \]

after evaluating the integral by re-expressing it as a gamma density function. Therefore, this ML estimator, $\hat{\sigma}_{ML}$, is unbiased.

e) From the delta method and c), we have
\[ \text{Var}(\hat{\sigma}_{MM}) \approx [g'(\mu_Y)]^2 \text{Var}(\bar{Y}) = [g'(\mu_Y)]^2 \frac{\text{Var}(Y)}{n}. \]

Substituting from expressions obtained above
\[ \text{Var}(\hat{\sigma}_{MM}) \approx \left\{ \frac{1}{2\sqrt{2}} (2\sigma^2)^{-\frac{1}{2}} \right\}^2 \frac{20\sigma^4}{n} = \frac{5\sigma^2}{4n}. \]

f) Because $X_1, \ldots, X_n$ are independent and identically distributed,
\[ \text{Var}(\hat{\sigma}_{ML}) = \text{Var}\left(\frac{1}{n} \sum_{i=1}^{n} |X_i|\right) = \frac{1}{n} \text{Var}\left(|X|\right). \]

But $E(|X|^2) = E(X^2) = 2\sigma^2$ from c) and $E(|X|) = \sigma$ from d), so we obtain
\[ \text{Var}(\hat{\sigma}_{ML}) = \frac{\sigma^2}{n}. \]

g) From the above results,
\[ \text{MSE}(\hat{\theta}_{ML}) = \frac{\sigma^2}{n} + 0 = \frac{\sigma^2}{n}, \]
and
\[ \text{MSE}(\hat{\theta}_{MM}) \approx \frac{5\sigma^2}{4n} + \left[ -\frac{5\sigma}{8n} \right]^2 = \frac{5\sigma^2}{4n} + \frac{25\sigma^2}{64n^2} = \frac{5\sigma^2}{4n} \]
to leading order (in $n$).

h) For this problem, we prefer the ML estimator because both:

- the bias of the MLE is smaller (in magnitude) than that of the MME (in fact, the MLE is unbiased).
- the variance of the MLE is smaller than that of the MME.