Solution to Problem 8.16

8.16. a) For the double exponential probability density function

$$f(x|\sigma) = \frac{1}{2\sigma}\exp\left(-\frac{|x|}{\sigma}\right),$$

the first population moment, the expected value of X, is given by

$$E(X) = \int_{-\infty}^{\infty} \frac{x}{2\sigma}\exp\left(-\frac{|x|}{\sigma}\right)\,dx = 0$$

because the integrand is an odd function ($g(-x) = -g(x)$). The first population moment does not depend on the unknown parameter σ, so it cannot be used to develop a method of moments estimator of σ. We go on to consider the second population moment:

$$E(X^2) = \int_{-\infty}^{\infty} \frac{x^2}{2\sigma}\exp\left(-\frac{|x|}{\sigma}\right)\,dx = 2\int_{0}^{\infty} \frac{x^2}{2\sigma}\exp\left(-\frac{x}{\sigma}\right)\,dx.$$

Re-expressing the last integral as the integral of a gamma density function leads to

$$E(X^2) = 2\sigma^2$$

as $\int_0^\infty \frac{\left(\frac{1}{\sigma}\right)^3}{\Gamma(3)} x^{3-1}\exp\left(-\frac{x}{\sigma}\right)\,dx = 1$. Solving for σ in terms of $\mu_2 = E(X^2)$ leads to

$$\sigma = \sqrt{\frac{1}{2}\mu_2}.$$

Therefore, the method of moments estimator of σ is given by

$$\hat{\sigma}_{MM} = \sqrt{\frac{1}{2}\hat{\mu}_2} = \sqrt{\frac{1}{2n}\sum_{i=1}^{n} X_i^2}.$$

b) The likelihood function is given by

$$L(\sigma) = \prod_{i=1}^{n} f(x_i|\sigma) = \prod_{i=1}^{n} \frac{1}{2\sigma}\exp\left(-\frac{|x_i|}{\sigma}\right).$$

To simplify, consider the log-likelihood function

$$\begin{aligned} l(\sigma) &= \sum_{i=1}^{n}\left[-\log(2\sigma) - \frac{|x_i|}{\sigma}\right] \\ &= -n\log 2 - n\log\sigma - \frac{1}{\sigma}\sum_{i=1}^{n}|x_i| \\ \Rightarrow l'(\sigma) &= -\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{n}|x_i|. \end{aligned}$$

It follows that

$$l'(\sigma) > 0 \Leftrightarrow \sigma < \frac{1}{n}\sum_{i=1}^{n}|x_i|$$

so it is clear that the (unique) solution to $l'(\sigma) = 0$ corresponds to a maximum and we obtain

$$\hat{\sigma}_{ML} = \frac{1}{n}\sum_{i=1}^{n}|X_i|.$$

c) If we let $Y_i = X_i^2$, we can write

$$\hat{\sigma}_{MM} = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}X_i^2} = \sqrt{\bar{Y}/2} = g(\bar{Y}),$$

where $g(y) = \sqrt{y/2}$. The second order approximation to the expected value of $\hat{\sigma}_{MM}$ is then is given by

$$E(\hat{\sigma}_{MM}) \cong g(\mu_{\bar{Y}}) + \frac{1}{2}g''(\mu_{\bar{Y}})\text{Var}(\bar{Y})$$

But $Y_1, \ldots, Y_n$ are independent and identically distributed (because $X_1, \ldots, X_n$ are), so $\mu_{\bar{Y}} = E(\bar{Y}) = E(Y)$ and $\text{Var}(\bar{Y}) = \text{Var}(Y)/n$, leading to the simplified expression

$$E(\hat{\sigma}_{MM}) \cong g(\mu_Y) + \frac{1}{2n}g''(\mu_Y)\text{Var}(Y).$$

From a), we obtain

$$\mu_Y = E(Y) = E(X^2) = 2\sigma^2.$$

Also

$$\text{Var}(Y) = \text{Var}(X^2) = E(X^4) - [E(X^2)]^2,$$

and

$$E(X^4) = \int_{-\infty}^{\infty} x^4 \frac{1}{2\sigma}\exp\left(-\frac{|x|}{\sigma}\right)dx = \frac{2}{2\sigma}\int_0^{\infty} x^4 e^{-\frac{x}{\sigma}}\,dx = 24\sigma^4,$$

after evaluating the integral by re-expressing it as a gamma density function. So

$$\text{Var}(Y) = 24\sigma^4 - (2\sigma^2)^2 = 20\sigma^4,$$

and the expression becomes

$$E(\hat{\sigma}_{MM}) \cong \sigma + \frac{10\sigma^4}{n}g''(2\sigma^2).$$

Finally, we evaluate the derivatives

$$g'(y) = \frac{1}{2\sqrt{2}}y^{-\frac{1}{2}} \quad \text{and} \quad g''(y) = -\frac{1}{4\sqrt{2}}y^{-\frac{3}{2}},$$

to obtain

$$E(\hat{\sigma}_{MM}) \cong \sigma - \frac{5\sigma}{8n},$$

and

$$\text{Bias}(\hat{\sigma}_{MM}) \cong -\frac{5\sigma}{8n}.$$

As is typical for reasonable estimators, the bias is of order $1/n$. So, for large values of n, the MSE of $\hat{\sigma}_{MM}$ will be dominated by the variance (because MSE = Variance + Bias^2).

d) Because $X_1, \ldots, X_n$ are independent and identically distributed,

$$E(\hat{\sigma}_{ML}) = E\left(\frac{1}{n}\sum_{i=1}^{n}|X_i|\right) = E\left(|X|\right).$$

But

$$E\left(|X|\right) = \int_{-\infty}^{\infty} \frac{|x|}{2\sigma}\exp\left(-\frac{|x|}{\sigma}\right)\,dx = 2\int_{0}^{\infty}\frac{x}{2\sigma}\exp\left(-\frac{x}{\sigma}\right)\,dx = \sigma,$$

after evaluating the integral by re-expressing it as a gamma density function. Therefore, this ML estimator, $\hat{\sigma}_{ML}$, is unbiased.

e) From the delta method and c), we have

$$\text{Var}(\hat{\sigma}_{MM}) \cong \left[g'(\mu_{\bar{Y}})\right]^2 \text{Var}(\bar{Y}) = \left[g'(\mu_Y)\right]^2 \frac{\text{Var}(Y)}{n}.$$

Substituting from expressions obtained above

$$\text{Var}(\hat{\sigma}_{MM}) \cong \left\{\frac{1}{2\sqrt{2}}(2\sigma^2)^{-\frac{1}{2}}\right\}^2 \frac{20\sigma^4}{n} = \frac{5\sigma^2}{4n}.$$

f) Because $X_1, \ldots, X_n$ are independent and identically distributed,

$$\text{Var}(\hat{\sigma}_{ML}) = \text{Var}\left(\frac{1}{n}\sum_{i=1}^{n}|X_i|\right) = \frac{1}{n}\text{Var}\left(|X|\right).$$

But $E(|X|^2) = E(X^2) = 2\sigma^2$ from c) and $E(|X|) = \sigma$ from d), so we obtain

$$\text{Var}(\hat{\sigma}_{ML}) = \frac{\sigma^2}{n}.$$

g) From the above results,

$$MSE(\hat{\theta}_{ML}) = \frac{\sigma^2}{n} + 0 = \frac{\sigma^2}{n},$$

and

$$MSE(\hat{\theta}_{MM}) \cong \frac{5\sigma^2}{4n} + \left[-\frac{5\sigma}{8n}\right]^2 = \frac{5\sigma^2}{4n} + \frac{25\sigma^2}{64n^2} = \frac{5\sigma^2}{4n} \quad \text{to leading order (in } n\text{)}.$$

h) For this problem, we prefer the ML estimator because both:

- the bias of the MLE is smaller (in magnitude) than that of the MME (in fact, the MLE is unbiased).
- the variance of the MLE is smaller than that of the MME.