Solution to Problem 8.50

8.50. See Lab # 6 for Reza's alternate solution to parts a), b) and c).

a) We need to evaluate the first population moment:

$$\mu_1 = E(X) = \int_0^\infty x f(x)\, dx = \int_0^\infty \frac{x^2}{\theta^2} e^{-\frac{x^2}{2\theta^2}}\, dx.$$

It is probably easiest (as almost always) to evaluate this integral by relating it to a familiar density function. The limits of integration are from 0 to ∞ and the integrand is of the form $x^2 e^{-cx^2}$ (where $c > 0$), so if we use the transformation $y = x^2$, this will be related to a gamma density function. Let $y = x^2 \Rightarrow x = \sqrt{y} \Rightarrow dx = \frac{1}{2} y^{-\frac{1}{2}} dy$, to obtain

$$\mu_1 = \int_0^\infty \frac{y}{\theta^2} e^{-\frac{y}{2\theta^2}} \times \frac{1}{2} y^{-\frac{1}{2}}\, dy = \frac{1}{2\theta^2} \int_0^\infty y^{\frac{1}{2}} e^{-\frac{y}{2\theta^2}}\, dy = \frac{1}{2\theta^2} \frac{\Gamma\left(\frac{3}{2}\right)}{\left(\frac{1}{2\theta^2}\right)^{\frac{3}{2}}} \int_0^\infty \frac{\left(\frac{1}{2\theta^2}\right)^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)} y^{\frac{3}{2}-1} e^{-\frac{1}{2\theta^2} y}\, dy$$

where we have included the constants required inside the integral sign to make the integrand exactly equal to a gamma density function (with $\alpha = 3/2$ and $\lambda = 1/2\theta^2$). This yields

$$\mu_1 = \sqrt{2\theta^2}\, \Gamma\left(\frac{3}{2}\right) = \sqrt{2\theta^2}\, \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \sqrt{2\theta^2}\, \frac{1}{2}\sqrt{\pi} = \sqrt{\pi/2}\ \theta.$$

Re-expressing this, we have $\theta = \sqrt{2/\pi}\ \mu_1$, so we obtain $\hat{\theta}_{MM} = \sqrt{2/\pi}\ \bar{X}$.

Alternatively, you could express the integral in terms of the normal density:

$$\mu_1 = \int_0^\infty \frac{x^2}{\theta^2} e^{-\frac{x^2}{2\theta^2}}\, dx = \frac{1}{2} \int_{-\infty}^\infty \frac{x^2}{\theta^2} e^{-\frac{x^2}{2\theta^2}}\, dx,$$

where the second step is allowed because the integrand is an even function of x. So we obtain

$$\mu_1 = \frac{1}{2\theta} \sqrt{2\pi} \int_{-\infty}^\infty \frac{x^2}{\sqrt{2\pi}\,\theta} e^{-\frac{x^2}{2\theta^2}}\, dx = \frac{1}{2\theta}\sqrt{2\pi}\ E(X^2),$$

where $X \sim N(0,\, \theta^2)$. This yields

$$\mu_1 = \frac{1}{2\theta}\sqrt{2\pi}\ \theta^2 = \sqrt{\pi/2}\ \theta,$$

as above.

b) The likelihood function is given by

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} \frac{x_i}{\theta^2} e^{-\frac{x_i^2}{2\theta^2}} \\ \Rightarrow l(\theta) &= \sum_{i=1}^{n} \log x_i - 2n \log \theta - \frac{1}{2\theta^2} \sum_{i=1}^{n} x_i^2 \\ \Rightarrow l'(\theta) &= -\frac{2n}{\theta} + \frac{1}{\theta^3} \sum_{i=1}^{n} x_i^2 \end{aligned}$$

It follows that

$$l'(\theta) > 0 \Leftrightarrow \theta^2 < \frac{1}{2n} \sum_{i=1}^{n} x_i^2,$$

so the maximum likelihood estimator (MLE) is given by $\hat{\theta}_{ML} = \sqrt{\frac{1}{2n} \sum_{i=1}^{n} X_i^2}$.

c) We have

$$\begin{aligned} l''(\theta) &= \frac{2n}{\theta^2} - \frac{3}{\theta^4} \sum_{i=1}^{n} x_i^2 \\ \Rightarrow E(l''(\theta)) &= \frac{2n}{\theta^2} - \frac{3n}{\theta^4} E(X^2). \end{aligned}$$

So we need to calculate

$$E(X^2) = \int_0^\infty \frac{x^3}{\theta^2} e^{-\frac{x^2}{2\theta^2}} \, dx.$$

Transform by letting $y = x^2$ (as above) to obtain

$$E(X^2) = \int_0^\infty \frac{y^{\frac{3}{2}}}{\theta^2} e^{-\frac{y}{2\theta^2}} \frac{1}{2} y^{-\frac{1}{2}} \, dy = \frac{1}{2\theta^2} \frac{\Gamma(2)}{\left(\frac{1}{2\theta^2}\right)^2} \int_0^\infty \frac{\left(\frac{1}{2\theta^2}\right)^2}{\Gamma(2)} y^{2-1} e^{-\frac{1}{2\theta^2} y} \, dy = \frac{1}{2\theta^2} \times (2\theta^2)^2 = 2\theta^2.$$

So

$$E(l''(\theta)) = \frac{2n}{\theta^2} - \frac{3n}{\theta^4} 2\theta^2 = -\frac{4n}{\theta^2},$$

and the asymptotic variance of the MLE is given by

$$\text{A.Var}(\hat{\theta}_{ML}) = -\frac{1}{E(l''(\theta))} = \frac{\theta^2}{4n}.$$

d) We have

$$\text{Var}(\hat{\theta}_{MM}) = \frac{2}{\pi} \text{Var}(\bar{X}) = \frac{2}{\pi} \frac{1}{n} \text{Var}(X) = \frac{2}{n\pi} [E(X^2) - (E(X))^2].$$

Substituting the values for $E(X)$ and $E(X^2)$ from above, we obtain

$$\text{Var}(\hat{\theta}_{MM}) = \frac{2}{n\pi} \left[2\theta^2 - \frac{2\pi}{4} \theta^2 \right] = \frac{\theta^2}{n} \left[\frac{4}{\pi} - 1 \right].$$

e)

$$\frac{\text{A.Var}(\hat{\theta}_{ML})}{\text{Var}(\hat{\theta}_{MM})} = \frac{1}{4\left(\frac{4}{\pi} - 1\right)} \cong 0.915 < 1.$$

Because $\text{A.Var}(\hat{\theta}_{ML}) < \text{Var}(\hat{\theta}_{MM})$, the MLE is more precise than the MME — at least for large values of n — so prefer to use the MLE. But, it is important to realize that this comparison is not complete because we have compared the **asymptotic variance of the MLE** to the **exact variance of the MME** and, further, we have ignored that fact that the MME is unbiased whereas the MLE is only asymptotically unbiased. (You might want to use the second order approximation provided by the delta method to evaluate $E(\hat{\theta}_{ML}) \cong \theta - \theta/8n \Rightarrow \text{Bias}(\hat{\theta}_{ML}) \cong -\theta/8n$.) For a more comprehensive comparison, we would need to carry out a simulation study where we simulate samples from the Rayleigh distribution and compare the performance of the MME and MLE in the simulated samples. We would simulate lots of samples – at least 1000, maybe more – from each combination of θ and n to be investigated. This would allow us to describe how the MSEs of the estimators compare as n increases (for fixed values of θ) and as θ changes (for fixed values of n). **Question:** How many samples are necessary to get a good description of the performance?