Solutions to Problems 8.2 and 8.4 (a-d)

8.2. To fit a Poisson distribution, we first have to estimate the parameter λ. With a Poisson model for the number of right turns during the 300 3-min intervals, we model the 300 counts as the observed values of 300 independent and identical Poisson random variables $X_1, \ldots, X_{300}$. Using either MM or ML, we estimate the value of the parameter λ by the average of those 300 realizations:

$$\hat{\lambda} = \frac{(0 \times 14) + (1 \times 30) + \cdots + (12 \times 1) + (13 \times 0)}{300} = \frac{1168}{300} = 3.893.$$

We then have $\hat{p}_i = \hat{\lambda}^i \, exp(-\hat{\lambda})/i!$ as the estimated probability that the number of right turns in any of the 300 3-min intervals is i, so the expected frequency is $300 \times \hat{p}_i$:

n	Observed Frequency	Estimated Probability	Expected Frequency
0	14	0.0204	6.1
1	30	0.0793	23.8
2	36	0.1544	46.3
3	68	0.2004	60.1
4	43	0.1951	58.5
5	43	0.1519	45.6
6	30	0.0986	29.6
7	14	0.0548	16.4
8	10	0.0267	8.0
9	6	0.0115	3.5
10	4	0.0045	1.3
11	1	0.0016	0.5
12	1	0.0005	0.2
13+	0	0.0002	0.1
	300	1.0000	300

As you can see, the observed and expected counts do not agree particularly well. A formal method for assessing the adequacy of the fit of the model to the data (Do the data provide convincing evidence that the Poisson model is inadequate?) can be based on Pearson's chi-square statistic (Section 9.5) or the Poisson dispersion test (Section 9.6).

8.4. We have $P(X = i) = 2\theta/3$, $\theta/3$, $2(1-\theta)/3$, and $(1-\theta)/3$ for $i = 0, 1, 2,$ and 3, respectively.

a) The expected value of X, the first population moment, is given by

$$\begin{aligned} E(X) &= \sum_{i=0}^{3} i\, P(X = i) \\ &= 0 \times 2\theta/3 + 1 \times \theta/3 + 2 \times 2(1-\theta)/3 + 3 \times (1-\theta)/3 \\ &= \frac{7}{3} - 2\theta. \end{aligned}$$

Solving for θ yields

$$\theta = \frac{7}{6} - \frac{E(X)}{2}.$$

Therefore, the method of moments estimator of θ is

$$\hat{\theta}_{MM} = \frac{7}{6} - \frac{\hat{\mu}_1}{2} = \frac{7}{6} - \frac{\bar{X}}{2}.$$

The observed value of the sample mean is $\bar{x} = 15/10 = 1.5$, so for this data set the value of the method of moments estimate is $\hat{\theta}_{MM} = \frac{5}{12} \cong 0.417$.

b) We can find the exact expression for $SE(\hat{\theta}_{MM})$, the standard error of the estimator $\hat{\theta}_{MM}$. We have

$$\mathrm{Var}(\hat{\theta}_{MM}) = \mathrm{Var}\left(\frac{7}{6} - \frac{\bar{X}}{2}\right) = \frac{1}{4}\mathrm{Var}(\bar{X}) = \frac{1}{4n}\mathrm{Var}(X).$$

From

$$\begin{aligned} E(X^2) &= \sum_{i=0}^{3} i^2\, P(X = i) \\ &= 0^2 \times 2\theta/3 + 1^2 \times \theta/3 + 2^2 \times 2(1-\theta)/3 + 3^2 \times (1-\theta)/3 \\ &= \frac{17}{3} - \frac{16}{3}\theta, \end{aligned}$$

we obtain

$$\mathrm{Var}(X) = \frac{17}{3} - \frac{16}{3}\theta - \left(\frac{7}{3} - 2\theta\right)^2 = \frac{2}{9} + 4\theta - 4\theta^2,$$

and hence

$$SE^2(\hat{\theta}) = \left(\theta(1-\theta) + \frac{1}{18}\right)/n.$$

For this data set, we have $\hat{\theta}_{MM} = \frac{5}{12}$, so the estimated standard error of $\hat{\theta}_{MM}$ for this data set is given by

$$\hat{SE}^2(\hat{\theta}_{MM}) = \left((\frac{5}{12})(1 - \frac{5}{12}) + \frac{1}{18}\right)/10 \cong 0.2986/10 = 0.02986,$$

or $\hat{SE}(\hat{\theta}_{MM}) \cong 0.173$.

c) If x_i denotes the observed value of X_i, then the likelihood function is

$$L(\theta) = \prod_{i=1}^{n} P(X_i = x_i),$$

and the log-likelihood function is

$$l(\theta) = \sum_{i=1}^{n} \log P(X_i = x_i).$$

Now each x_i is either $0, 1, 2$, or 3. Let n_j be the number of times the value j is observed ($\sum_{j=0}^{3} n_j = n$). Then the log-likelihood function becomes

$$\begin{aligned} l(\theta) &= \sum_{j=0}^{3} n_j \log P(X=j) \\ &= n_0 \log \frac{2}{3}\theta + n_1 \log \frac{1}{3}\theta + n_2 \log \frac{2}{3}(1-\theta) + n_3 \log \frac{1}{3}(1-\theta) \\ &= c + (n_0+n_1)\log\theta + (n_2+n_3)\log(1-\theta), \end{aligned}$$

where c does not depend upon θ. Differentiating once with respect to θ yields

$$l'(\theta) = \frac{n_0+n_1}{\theta} - \frac{n_2+n_3}{(1-\theta)} = \frac{(n_0+n_1)(1-\theta) - (n_2+n_3)\theta}{\theta(1-\theta)}.$$

It follows that

$$l'(\theta) > 0 \Leftrightarrow \theta < (n_0+n_1)/n,$$

so it is clear that the (unique) solution of $l'(\theta) = 0$ given by $\theta = (n_0+n_1)/n$ corresponds to a maximum; that is,

$$\hat{\theta}_{ML} = (N_0+N_1)/n.$$

(Alternately, it is easy to show that $l''(\theta) < 0$ for all values of θ leading to the same conclusion.)

For this data set, we have $n_0 = 2, n_1 = 3, n_2 = 3$ and $n_3 = 2$, so we obtain the value of the ML estimator (that is, **the ML estimate for this data set**) to be $\hat{\theta}_{ML} = 5/10 = 0.5$.

Note: In the above we started with the likelihood of the original $X_1, \ldots, X_n$. As the X_i's take on only 4 possible values, this is a multinomial situation (Section 8.5.1) and you could equally well start with the likelihood of the derived random variables N_0, N_1, N_2, and N_3 ($N_0+N_1+N_2+N_3 = n$). You should check that you get the same expression for $\hat{\theta}_{ML}$ either way.

d) The variance of the MLE is given by

$$\mathrm{Var}(\hat{\theta}_{ML}) = \mathrm{Var}\left(\frac{N_0+N_1}{n}\right) = \frac{1}{n^2}\mathrm{Var}(N_0+N_1).$$

But $N_0+N_1 \sim \mathrm{Binom}(n, p_0+p_1)$, where $p_0+p_1 = \theta/3 + 2\theta/3 = \theta$, so

$$\mathrm{Var}(\theta_{ML}) = n\theta(1-\theta)/n^2 = \theta(1-\theta)/n$$

and hence

$$SE(\hat{\theta}_{ML}) = \sqrt{\theta(1-\theta)/n}\,.$$

Subsituting in $\hat{\theta}_{ML} = 0.5$ yields the estimated standard error as

$$\hat{SE}(\hat{\theta}_{ML}) = \sqrt{0.025} \cong 0.158.$$

Note: You could instead evaluate $\mathrm{Var}(N_0+N_1) = \mathrm{Var}(N_0) + \mathrm{Var}(N_1) + 2\mathrm{Cov}(N_0, N_1)$. $\mathrm{Var}(N_0)$ and $\mathrm{Var}(N_1)$ are immediate as $N_0 \sim \mathrm{Binom}(n, p_0)$ and $N_1 \sim \mathrm{Binom}(n, p_1)$, but you also need to know more about the properties of the multinomial distribution to complete this evaluation; namely that $\mathrm{Cov}(N_0, N_1) = -np_0p_1$.