Solutions to Some Problems on "Delta Method"

4.100. We use the delta method for the approximation. We have

$$Y = g(X) \approx g(\mu_X) + (X - \mu_X)g'(\mu_X)$$

for the first order, and

$$Y = g(X) \approx g(\mu_X) + (X - \mu_X)g'(\mu_X) + \frac{1}{2}(X - \mu_X)^2 g''(\mu_X)$$

for the second order. The first order approximation is a linear function of X so leads immediately to

$$E(Y) \approx g(\mu_X) \quad \text{and} \quad \mathrm{Var}(Y) \approx [g'(\mu_X)]^2 \ \mathrm{Var}(X).$$

The second order approximation is a quadratic function of X that yields the approximation:

$$E(Y) \approx g(\mu_X) + \frac{1}{2}\ g''(\mu_X)\ \mathrm{Var}(X).$$

For the case $g(x) = 1/x$, we have $g'(x) = -1/x^2$ and $g''(x) = 2/x^3$, so the first order approximation becomes:

$$E(Y) \approx 1/\mu_X \quad \text{and} \quad \mathrm{Var}(Y) \approx \sigma_X^2/\mu_X^4.$$

Similarly, the second order approximation yields:

$$E(Y) \approx 1/\mu_X + \sigma_X^2/\mu_X^3.$$

With $X \sim U[10, 20]$, we have $E(X) = \mu_X = 15$ and $\mathrm{Var}(X) = \sigma_X^2 = 25/3$, so the first order approximations become:

$$E(Y) \approx 1/15 \approx 0.0667 \quad \text{and} \quad \mathrm{Var}(Y) \approx 25/(3)(15^4) = 1/6075 \approx 0.000165.$$

while the second order approximation yields

$$E(Y) \approx 1/15 + 25/(3)(15^3) \approx 0.0691.$$

To find the exact values of $E(Y)$ and $\mathrm{Var}(Y)$, we can either calculate the mean and the variance of $Y = 1/X$ by evaluating $E(X^{-1})$ and $E(X^{-2})$ directly from the distribution of X, or we can first find the density of Y and then use that density to evaluate the mean and the variance of Y. The former approach is easier but, for purposes of illustration, we use the latter. The change-of-variable formula (or, as an alternative, see Example D on p. 62) yields

$$f_Y(y) = \frac{1}{10y^2} \quad \text{for} \quad y \in \left[\frac{1}{20}, \frac{1}{10}\right].$$

Then we obtain

$$E(Y) = \int_{\frac{1}{20}}^{\frac{1}{10}} \frac{1}{10y}\,dy = \frac{1}{10}\ln y\Big|_{\frac{1}{20}}^{\frac{1}{10}} \approx 0.0693,$$

and

$$E(Y^2) = \int_{\frac{1}{20}}^{\frac{1}{10}} \frac{1}{10}\,dy = \frac{1}{10}y\Big|_{\frac{1}{20}}^{\frac{1}{10}} = \frac{1}{200},$$

from which we obtain $\mathrm{Var}(Y) = 1/200 - (0.0693)^2 = 0.000195$.

Thus, the first and second order approximations to $E(Y)$ have relative errors of -3.8% and -0.3%, respectively, so going to the second order approximation provides considerable improvement. The first order approximation to $\mathrm{Var}(Y)$ has a relative error of about -15.8% but if expressed on the more relevant standard deviation (SD) scale, the relative error is about -8.2%. Still not a great approximation, but maybe adequate for some purposes. The approximation to the variance could be improved by using the second order approximation instead of the first order approximation, but that leads to a complicated expression also involving the 3^{rd} and 4^{th} moments of X.

4.103. We have the relationship $V = \pi D^3/6$, that is, $V = g(D)$, where $g(d) = \pi d^3/6$. We are also given $E(D) = \mu_D = 2$ mm and $\mathrm{SD}(D) = \sigma_D = 0.01$ mm. Using the first-order approximation provided by the delta method, since $g'(d) = \pi d^2/2$, we obtain

$$\mathrm{Var}(V) \cong [g'(\mu_D)]^2\ \sigma_D^2 = [\pi\mu_D^2/2]^2\ \sigma_D^2,$$

or

$$\mathrm{SD}(V) \cong \pi\mu_D^2\sigma_D/2.$$

Thus,

$$\mathrm{SD}(V) \approx 2\pi\ \sigma_D = 0.02\pi \text{ mm}.$$

4.104. Let our measurements be given by R and Θ, with means given by the (unknown) true values r and θ, respectively. That is, we can think of the measurements arising as $R = r + \epsilon_R$ and $\Theta = \theta + \epsilon_\Theta$, where ϵ_R and ϵ_Θ are the measurement errors. We are told that R and Θ are independent; that is, the two measurement errors are independent. Then the estimated altitude is a function of the two measurements, $Y = g(R, \Theta) = R\sin\Theta$. Set $\mu = (r,\ \theta)$.

a) Using the first-order delta method approximation,

$$\mathrm{Var}(Y) \cong \sigma_R^2\left[\frac{\partial g(\mu)}{\partial R}\right]^2 + \sigma_\Theta^2\left[\frac{\partial g(\mu)}{\partial \Theta}\right]^2 + 2\sigma_{R\Theta}\left[\frac{\partial g(\mu)}{\partial R}\right]\left[\frac{\partial g(\mu)}{\partial \Theta}\right],$$

where

$$\frac{\partial g(R,\Theta)}{\partial R} = \sin\Theta \qquad \text{and} \qquad \frac{\partial g(R,\Theta)}{\partial \Theta} = R\cos\Theta.$$

So

$$\frac{\partial g(\mu)}{\partial R} = \sin\theta \qquad \text{and} \qquad \frac{\partial g(\mu)}{\partial \Theta} = r\cos\theta.$$

Further, $\sigma_{R\Theta} = 0$ as R and Θ are independent. Substituting yields

$$\mathrm{Var}(Y) \cong \sigma_R^2\,(\sin\theta)^2 + \sigma_\Theta^2\,(r\cos\theta)^2\,.$$

b) If R is held fixed (at the value r), then Y is a function of only one random variable Θ. Applying the delta method (or, more directly, substituting $\sigma_R^2 = 0$ into the expression above), we then have

$$\mathrm{Var}(Y) \cong \sigma_\Theta^2 \, (r \cos\theta)^2 \, .$$

This function of θ takes on its largest value when $|\cos\theta\,|$ is maximized; that is, when θ is either 0 or π.