Solution to Problem 5.5

5.5. If $Y \sim \text{Binom}(n, p)$, then the moment generating function of Y is given by

$$M_Y(t) = \Big(1 - p + pe^t\Big)^n = \Big(1 + p(e^t - 1)\Big)^n.$$

Substituting $p = \lambda/n$ leads to

$$M_Y(t) = \left(1 + \frac{\lambda(e^t - 1)}{n}\right)^n.$$

Because

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x$$

for any (fixed) value of x, we obtain:

$$\lim_{n\to\infty}\left(1 + \frac{\lambda(e^t - 1)}{n}\right)^n = \exp\Big(\lambda(e^t - 1)\Big),$$

which is the mgf of a Poisson(λ) random variable.

This result implies that for large n and small p (the result is obtained by assuming that $np = \lambda$ does not change as $n \to \infty$, so as n is getting large, p must become small), the binomial random variable Y can be approximated as a Poisson random variable with parameter $\lambda = np$. That is, for "rare" events, the number of "successes" in n coin tosses can be approximated as the number of events for a Poisson($\lambda = np$) random variable. The result is useful because the Poisson probabilities are often much easier to calculate than the binomial probabilities.

Note added later: Reza has pointed out that the question asks you to show the result "as $n \to \infty$, $p \to 0$, and $np \to \lambda$", **not** "as $n \to \infty$ and $p \to 0$ in such a way that $np = \lambda$", as is used in the argument above. For practical purposes, the question could equally well have been asked in the second form but to solve the problem as stated, you cannot substitute $p = \lambda/n$ into the expression for $M_Y(t)$ because that condition is not given as part of the statement of the problem.

One way to fix the argument is to instead set $np = \lambda + \epsilon_n$, where $\epsilon_n \to 0$ as $n \to \infty$ so that we have $np \to \lambda$ as $n \to \infty$ as given (rather than $np = \lambda$ as $n \to \infty$). We then obtain

$$M_Y(t) = \left(1 + \frac{(\lambda + \epsilon_n)(e^t - 1)}{n}\right)^n = \left(1 + \frac{\lambda(e^t - 1) + \epsilon_n^*}{n}\right)^n,$$

where $\epsilon_n^* = \epsilon_n(e^t - 1) \to 0$ as $n \to \infty$. Now we use the slightly more general result that

$$\lim_{n\to\infty}\left(1 + \frac{x + \epsilon_n^*}{n}\right)^n = e^x$$

for any (fixed) value of x as long as $\epsilon_n^* \to 0$ as $n \to \infty$ to obtain the desired result.