Question 9.1
a)

Let us denote the head by 1 and the tail by zero. Let X be a random variable such that $P\{X = 1\} = \theta$ and $P\{X = 0\} = 1 - \theta$. We are given a sample $X_1, \cdots, X_{10}$ independent and identically distributed as X. Then $\sum_{i=1}^{10} X_i \sim \text{Binomial}(\theta, 10)$

$$
\begin{aligned}
\alpha = P\{\text{Rejecting } H_0 | H_0\} = P\{\sum_{i=1}^{10} X_i = 10 \text{ or } 0\} = \\
P\{\sum_{i=1}^{10} X_i = 10\} + P\{\sum_{i=1}^{10} X_i = 10\} = \\
(\frac{1}{2})^{10} + (\frac{1}{2})^{10} = 2(\frac{1}{2})^{10} = 0.002
\end{aligned}
$$

b)

$$
\begin{aligned}
\text{Power} = P\{\text{Rejecting } H_0 | \theta = 0.1\} = \\
P\{\sum_{i=1}^{10} X_i = 10 \text{ or } 0\} = \\
P\{\sum_{i=1}^{10} X_i = 10\} + P\{\sum_{i=1}^{10} X_i = 10\} = \\
\frac{1}{10}^{10} + \frac{9}{10}^{10} = 0.349
\end{aligned}
$$

Question 9.12

The likelihood is given by

$$f(x_1, \cdots, x_n | \theta) = \theta^n \exp(-\theta n \bar{X})$$

The ratio test is given by

$$\Lambda = \frac{\max_{\theta \in \omega_0}[lik(\theta)]}{\max_{\theta \in \Omega}[lik(\theta)]} < c_1$$

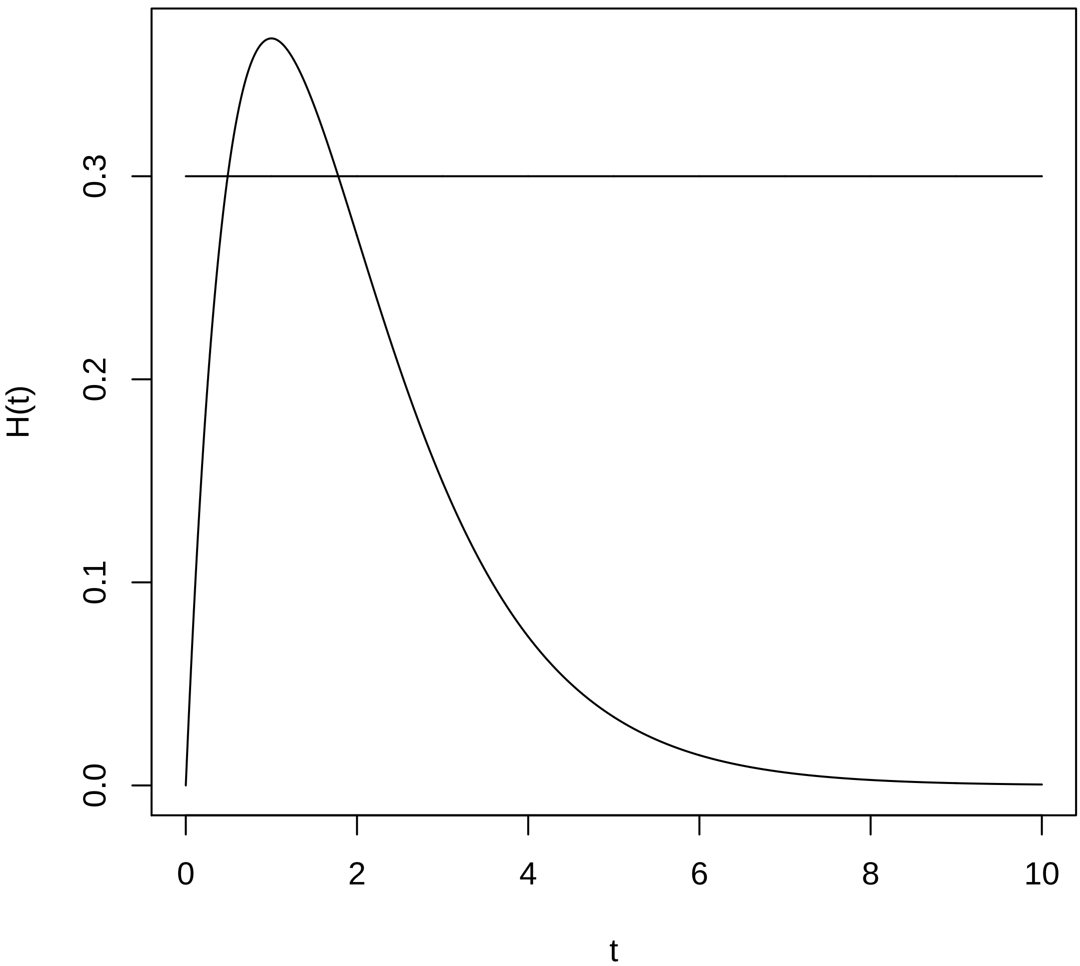
The max for the denominator is obtained by the MLE which is equal to $\frac{1}{\bar{X}}$. The max in the numerator is even easier to find. We are maximizing over a set with only one element, namely θ_0 and hence the max happens at θ_0. Hence, we have

$$\frac{\theta_0^n \exp(-\theta_0 n\bar{X})}{(\frac{1}{\bar{X}})^n \exp(-n)} < c_1 \Rightarrow$$
$$\bar{X}\exp(-\theta_0\bar{X}) < (\frac{c_1\exp(-n)}{\theta_0^n})^{\frac{1}{n}} = c$$

Question 9.13

a)
By 9.12, for $\theta_0 = 1$,
$$\bar{X}\exp(-\bar{X}) < c$$
as the rejection region. Define,
$$H(t) = t\exp(-t) < c,\ t > 0$$
Then
$$H'(t) = \exp(-t) - t\exp(-t) = 0$$
And so H has a local extremum at 1. Also, H' is decreasing after 1 and increasing after 1. The limit of H at zero and infinity is zero as well. Based on this we can plot H as given in the figure. The rejection region is obtained by intersecting the line $y = c$. And so the the rejection region will have the form
$$\{\bar{X} < x_0\} \cup \{\bar{X} > x_1\}$$
b) Simple replacement.
c)
One way to show that the summation of exponentials is gamma is using the moment generating functions.

For any given c, we can find x_0 and x_1 using part a). Note that although the equation does not have explicit solutions. We can find the corresponding x_0 and x_1 using approximation methods or a computer code. Once we find those values, we can compute



H(t)
0.0
0.1
0.2
0.3
0
2
4
6
8
10
t

$$P(\{\bar{X} < x_0\} \cup \{\bar{X} > x_1\})$$

Since the distribution of $\bar{X}$ is known. We are interested in a c that makes this probability equal to the given $\alpha = 0.05$. Hence, we can write a computer code to try different $c's$ and find the appropriate one.