Question 2.58
To find the density function for $\sqrt{U}$, we find the distribution function first and differentiate it.

$$F_{\sqrt{U}}(t) = P\{\sqrt{U} < t\} = \begin{cases} P\{0 < U < t^2\} & 0 \le t \le 1 \\ 0 & t < 0 \\ 1 & t > 1 \end{cases}$$

Hence, $f_{\sqrt{U}}(t) = 2t \quad 0 \le t \le 1$

Question 3.24
$P \sim U_{[0,1]}$ and $X|P \sim B(p)$ Hence,

$$f_P(p) = 1,\ 0 < p < 1 \ \text{ and } \ f_{X|P}(x|p) = \begin{cases} p & x = 1 \\ 1 - p & x = 0 \end{cases}$$

Let us compute $f_X(x)$ first:

$$f_X(1) = P_X\{X = 1\} = \int f_{X|P}(x = 1|p) f_P p dp = \int_0^1 p dp = \frac{1}{2}$$

But

$$f_X(0) = P_X\{X = 0\} = 1 - P_X\{X = 1\} = \frac{1}{2}$$

Now

$$f_{P|X}(p|x) = \frac{f_{P,X}(p,x)}{f_X(x)} = 2 f_{X|P}(x,p) f_P(p) = \begin{cases} 2p & x = 1, 0 < p < 1 \\ 2(1-p) & x = 0, 0 < p < 1 \end{cases}$$

Question 3.43
Let $Z = X + Y$ then since X, Y are independent to obtain $f_Z z$ we use the convolution integral of f_X and f_Y.

$$f_Z(z) = \int_R f_X(x) f_Y(z - x) dx = \int_{S(z)} 1 dx + \int_{S(z)'} 0 dx$$

$f_X(x) f_Y(z-x)$ is either 1 or 0. We have denote the region that $f_X(x) f_Y(z-x) = 1$ by $S(z)$ and the compliment of this region by $S(z)'$. To compute the integral we only need to find $S(z)$ for different values of z.

$$S(z) = \{x|\ f_X(x) = 1 \text{ and } f_Y(z-x) = 1\} =$$
$$\{x|\ 0 \leq x \leq 1 \text{ and } 0 \leq z - x \leq 1\} =$$
$$\{x|\ 0 \leq x \leq 1,\ x \leq z \text{ and } z - 1 \leq x\}$$

We consider four cases of z

- $z < 0 :\Rightarrow z - x < 0$ (since x is nonnegative.) $\Rightarrow S(z) = \phi$.
- $0 < z < 1$: since $x \leq z \leq 1 \Rightarrow S(z) = (0, z)$
- $1 < z < 1$: since, $0 < z - 1 \leq x \Rightarrow S(z) = (z - 1, 1)$
- $z > 2 \Rightarrow S(z) = \phi$. (Similar to the first case)

Now it is easy to compute the integral

$$f_Z(z) = \begin{cases} 0 & z < 0 \\ z & 0 \leq z \leq 1 \\ 2 - z & 1 < z \leq 2 \\ 0 & z > 2 \end{cases}$$

Question 4.96

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}M(s,t)|_{(s,t)=(0,0)} =$$
$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}E\{e^{sX+tY}\}|_{(s,t)=(0,0)} =$$
$$\frac{\partial}{\partial t}E\{Xe^{sX+tY}\}|_{(s,t)=(0,0)} =$$
$$E\{XYe^{sX+tY}\}|_{(s,t)=(0,0)} = E\{XY\}$$