Question 1.

Suppose $X_i \sim N(\mu, \sigma^2)$ $i = 1, \cdots, n$. Show how can $P\{a < \bar{X} < b, c < S^2 < d\}$ be obtained only using the normal and χ^2_k for an appropriate k.
Solution.
First of all note that $\bar{X}$ and S^2 are independent. Hence,

$$P\{a < \bar{X} < b, c < S^2 < d\} = P\{a < \bar{X} < b\}P\{c < S^2 < d\}$$

We are going to compute the two terms on the right hand side separately

$$P\{a < \bar{X} < b\} = P\{\frac{a-\mu}{\sigma} < \frac{\bar{X}-\mu}{\sigma} < \frac{b-\mu}{\sigma}\} = P\{\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\}$$

Now we can read the standard normal table for this one.
To compute the second one note that $(n-1)S^2/\sigma^2 \;\sim\; \chi^2_{n-1}$.

$$P\{c < S^2 < d\} = P\{(n-1)c/\sigma^2 < (n-1)S^2/\sigma^2 < (n-1)d/\sigma^2\}$$

Now, we are ready to read the χ^2_{n-1} table.

Question 2 (8.50)

Solution.
a.
To use the method of moments, we need to find the moments first. Since, there is only one parameter to estimate one moment should suffice.

$$\mu = \int_0^\infty x \frac{x}{\theta^2} e^{-x^2/(2\theta^2)} dx$$

To compute the integral, we use integration by part with $v = x, w =$

$\frac{x}{\theta^2}e^{-x^2/(2\theta^2)} \Rightarrow w = -e^{-x^2/2(\theta^2)}$.

$$\mu = \int_0^\infty x\frac{x}{\theta^2}e^{-x^2/(2\theta^2)} =$$

$$v.w|_0^\infty - \int_0^\infty 1.(-e^{-x^2/(2\theta^2)}) =$$

$$0 + \int_0^\infty (e^{-x^2/(2\theta^2)}) = 1/2\int_{-\infty}^\infty (e^{-x^2/(2\theta^2)}) = \sqrt{\frac{\pi}{2}}\theta\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}\sqrt{2}\theta}(e^{-x^2/(2\theta^2)}) = \sqrt{\frac{\pi}{2}}\theta \times 1$$

The last equality is concluded using normal distribution for $\mu = 0$ and $\sigma^2 = \theta^2$. Solving for the desired parameter θ gives,

$$\theta = \mu\sqrt{\frac{2}{\pi}}$$

Now that we have found μ, we can replace everything with the hat version. Hence,

$$\hat{\theta} = \hat{\mu}\sqrt{\frac{2}{\pi}}.$$

b.

To use MLE, firstly, we need to compute the joint distribution.

$$f(x_1, \cdots, x_n|\theta) = \prod_{i=1}^n f(x_i|\theta) = (\prod_{i=1}^n x_i)\frac{1}{\theta^2}e^{(-\sum x_i^2)/(2\theta^2)}$$

Treating $x_i's$ as constant and θ as a variable, we get the likelihood function

$$L(\theta|x_1, \cdots, x_n) = (\prod_{i=1}^n x_i)\frac{1}{(\theta^2)^n}e^{(-\sum x_i^2)/(2\theta^2)}$$

To get the MLE, we need to maximize the above function with respect to θ. If we let $\lambda = \theta^2$, $\lambda > 0$, we can maximize for λ first and solve for θ. The answer will be $\theta = \sqrt{\lambda}$. Also, note that $(\prod_{i=1}^n x_i)$ is a positive constant and so for simplicity, we can eliminate it from the maximization process. Let $d = -(\sum x_i^2)$. Based on this discussion, we want to maximize

$$g(\lambda) = \frac{e^{d/(2\lambda)}}{\lambda^n}, \quad 0 < \lambda < \infty$$

To examine if this function have a global maximum and find it, we have to find the critical points and also evaluate the function on the boundaries the function is defined. 0 and ∞ are the boundaries, the limit of g as $\lambda \to 0, \infty$ are both zero.

Critical points are the points where the derivative vanishes or does not exist.

$$g'(\lambda) = 0 \Rightarrow \frac{-n}{\lambda^{n+1}} e^{d/\lambda^2} - \frac{d}{2\lambda^2} e^{d/\lambda^2} \frac{1}{\lambda^n} = 0$$

Hence,

$$\lambda = \frac{-d}{2n} \Rightarrow \theta = \sqrt{\frac{-d}{2n}} = \frac{\sqrt{\sum_{i=1}^n x_i^2}}{2n},$$

is the only critical point. To check if this is actually the global maximum, we need to plug in the value to see if we get a bigger value than the limits on the boundaries (zero and infinity). This is true obviously because the limit on the boundaries is zero and the value of the function for this is positive.
c.

By the asymptotic theory, we know that the variance of the MLE estimator is given by $\sqrt{\frac{1}{nI(\theta_0)}}$, where θ_0 is the true value of the parameter.

$$\begin{aligned}
I(\theta) = -E\{\frac{\partial^2}{\partial\theta^2} \log f(x|\theta)\} = \\
-E\{\frac{\partial^2}{\partial\theta^2}(\log x - 2log\theta - \frac{x^2}{2\theta^2})\} = \\
-E\{\frac{\partial}{\partial\theta}(\frac{-2}{\theta} - \frac{x^2}{2}(\frac{-2}{\theta^3}))\} = \\
-E\{\frac{2}{\theta^2} - 3\frac{x^2}{\theta^4}\} = \\
-\frac{2}{\theta^2} + \frac{3}{\theta^4}E\{X^2\}
\end{aligned}$$

But using integration by part we get

$$E\{X^2\} = 2\theta^2$$

Hence,

$$I(\theta) = \frac{4}{\theta^2}$$

and finally $Var(\hat{\theta}) = \sqrt{\frac{1}{nI(\theta_0)}} = \frac{\theta_0}{2\sqrt(n)}$.