Question 8.27

a.
Let T denote the lifetime of the electronic device. Then by assumption, the density function of T is given by

$$f(t|\tau) = (1/\tau)\exp(-t/\tau),\ t \geq 0$$

Hence,

$$F_T(t) = 1 - \exp(-\tau t)$$

We have 5 independent devices with lifetimes $T_1, \cdots, T_5$ and we are only given the minimum of those $V = \min_{i=1}^{5} T_i$, to find the likelihood function of τ, we need to find the density function for V. Suppose, v be the observed minimum and f_V denote the density function of V, then by example 3.7.A, $f_V(v) = 5(1/\tau)\exp(\frac{-5v}{\tau}),\ v > 0$, which is the likelihood for τ as well.
b.
To get the MLE, we need to maximize the log likelihood on $(0, \infty)$. We first check the boundary points. For both zero and ∞, we get zero for the likelihood. Hence, none of them can be the maximum as the function is positive otherwise. Let us find the critical values

$$(\log f_V(v))' = (\log 5 - \log\tau + \frac{-5v}{\tau})' = 0$$

Hence

$$\frac{-1}{\tau} + \frac{5v}{\tau^2} = 0$$

This gives $\hat{\tau} = 5V$.
c.
It is easy in this case to find the distribution for the MLE estimate. We know that $\hat{\tau} = 5V$ where $f_V(v) = 5(1/\tau_0)\exp(\frac{-5v}{\tau_0}),\ v > 0$, where τ_0 is the true parameter. We know that if $Y = g(X)$, then $f_Y(y) = f_X(g^{-1}(y))|(g^{-1})'(y)|$. Here $X = V$ and $g(V) = 5V$. Hence, $(g^{-1})(\tau) = 1/5\tau$.

$$f_{\hat{\tau}}(\tau) = \frac{1}{\tau_0}\exp(\frac{-\tau}{\tau_0}),\ \tau > 0$$

Note that, we can derive this directly and not using the formula given above.

d.

By part (c) the mle is distributed as exponential with parameter $\lambda = \frac{1}{\tau_0}$. Hence, $\text{var}(\hat{\tau}) = \lambda^{-2} = \tau_0^2 \Rightarrow \text{std} = \tau_0$.