Final Exam
17 December 2004, 8:30 – 11:00
Time allowed : 150 minutes.

Authorized material :

- One letter-size cheat sheet (2-sided).
- One scientific calculator without wireless communication feature.
- Probability tables (provided).

Instructions :

- The exam has 9 pages including this one. Page 9 is blank to provide extra space.
- Answer all 7 questions; the total number of points is 100.
- This exam is worth 35% of the term. You must pass this exam to pass the course.
- Write legibly; give complete solutions. Marks may be taken away for unclear solutions.
- You can use the back side of the sheets as drafts. If you use it for writing answers, indicate it clearly.

Last Name : ______________________________

First Name : ______________________________

Student Number : ______________________________

Signature : ______________________________

Question 1

A single observation from a discrete distribution is observed. Two simple hypotheses are considered.

$$\begin{aligned} H_0 &: X \text{ follows the distribution } P_{H_0} \\ H_1 &: X \text{ follows the distribution } P_{H_1} \end{aligned}$$

where the two distributions are as follow

x	0	1	2	3
$P_{H_0}(X=x)$	0.80	0.10	0.01	0.09
$P_{H_1}(X=x)$	0.20	0.05	0.65	0.10

Consider the test that rejects H_0 if $X = 2$.

a) [5 pts] What is the level of this test?

b) [5 pts] What is the probability of a type II error?

Question 2

Consider a sample $X_1, \ldots, X_n$ from a normal distribution with mean μ and unknown variance σ^2.

a) [8 pts] Construct a $100(1-\alpha)\%$ confidence interval for the mean of the population. Use an exact pivotal quantity. You do not need to prove the distribution of the pivotal quantity, but you must indicate clearly what it is.

b) [2 pts] Suppose a sample of size 20 led to the 90% confidence interval $\mu \in [0.32, 1.41]$. What could you conclude about the hypotheses below? Indicate the level of your test.

$$\begin{aligned} H_0 &: \mu = 0 \\ H_1 &: \mu \neq 0 \end{aligned}$$

Question 3

Answer the following two questions by a short paragraph.

a) [5 pts] What is a Q-Q plot used for?

b) [5 pts] Describe the placebo effect and how it affects the design of an experiment. Use an example to illustrate your explanation.

Question 4

A sample of size n comes from a log-normal distribution with an unknown parameter μ and with $\sigma^2 = 1$.

$$f(x) = \begin{cases} \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma x} e^{-\frac{\{\log(x)-\mu\}^2}{2\sigma^2}} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

a) [10 pts] Find the MLE of μ.

b) [10 pts] Derive the generalized likelihood ratio test of level α for the hypotheses

$$\begin{aligned} H_0 &: \mu = 0 \\ H_1 &: \mu \neq 0. \end{aligned}$$

Question 5

a) [10 pts] Calculate the moment generating function of a gamma distribution with parameters α and λ.

b) [10 pts] Consider a sample $X_1, \ldots, X_n$ from a gamma with parameters α and λ. Using the MGF, prove that if $2n\alpha$ is an integer, then

$$Y = 2\lambda \sum_{i=1}^{n} X_i$$

follows a chi-square distribution with $2n\alpha$ degrees of freedom.

Question 6

A Bayesian analysis is conducted on a sample of size n believed to come from a Poisson distribution with parameter θ. The prior knowledge on θ is summarized by an exponential distribution with mean $1/\lambda$.

a) [10 pts] What is the posterior distribution of θ?

b) [5 pts] What estimate of θ minimizes the loss function $\ell(\theta - \hat{\theta}) = (\theta - \hat{\theta})^2$?

c) [5 pts] You must select a banking plan. Two options are available. In option A, you pay \$10 per month for unlimited banking. In plan B, the standard rate is \$12 per month, but you may qualify for a special rebate. If your weekly number of transactions follows a Poisson distribution, a careful study of plan B shows that as long as the mean of that Poisson does not exceed 0.658, you will qualify for a \$9 monthly rebate on your plan. Otherwise, you need to pay the full rate. A priori, you believe that you do an average of 1 transaction per week. You choose randomly 15 weeks from last year and realize you made 7 transaction during that period. What plan should you choose?

Question 7

[10 pts] A sample of size n comes from an inverse-gaussian distribution with $\lambda = 1$. Find the most powerful test of level α to confront the hypotheses

$$\begin{aligned} H_0 &: \mu = 1 \\ H_1 &: \mu = \mu_1. \end{aligned}$$

Note that the density of the inverse-gaussian is

$$f(x) = \begin{cases} \left(\frac{\lambda}{2\pi x^3}\right)^{\frac{1}{2}} e^{-\frac{\lambda}{2\mu^2 x}(x-\mu)^2} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

with $\mu > 0$ and $\lambda > 0$. The following moments may be useful:

$$\mathrm{E}(X) = \mu\,, \qquad \mathrm{var}(X) = (\mu\lambda)^2.$$