Midterm I - Solution

Question 1.

(a) FALSE. X and Y are independent if

$$P(X=x, Y=y) = P(X=x)\,P(Y=y) \qquad \forall x,y.$$

but

$$P(X=2, Y=3) = 0 \neq P(X=2)P(Y=3) = \frac{1}{6}\cdot\frac{5}{36} = \frac{5}{216}.$$

(b) FALSE. A and B cannot be independent if they are disjoint.

A, B disjoint $\Rightarrow$ $A \cap B = \emptyset$

$$\begin{array}{l} P(A\cap B) = 0 \\ \quad\;\, \nparallel \\ P(A)P(B) > 0 \end{array}$$

(c) FALSE. $P(A|B) = \dfrac{P(A\cap B)}{P(B)} = \dfrac{P(A)}{P(B)} > P(A)$ since $P(B) = 1 - P(B^c) < 1.$

(d) FALSE. $f(x)$ is negative for $x \in [0, 1/\sqrt{6}]$.

(e) TRUE. $E(X) = E(E(X|Y)) = E(Y/2) = 1/2\lambda$.

Question 2.

(a)

$$\text{cov}(U,V) = \text{cov}(Z+X,\ Z-Y)$$

$$= \text{cov}(Z,Z) + \text{cov}(X,Z) - \text{cov}(Z,Y) - \text{cov}(X,Y).$$

$$= \sigma_Z^2 + 0 - 0 - 0.$$ (zeros are by independence).

$$= \sigma_Z^2$$

$$\text{var}(U) = \text{var}(Z+X) = \text{var}(Z) + \text{var}(X)$$ since $X \perp\!\!\!\perp Z$.

$$= \sigma_Z^2 + \sigma_X^2$$

$$\text{var}(V) = \text{var}(Z-Y) = \text{var}(Z) + (-1)^2\,\text{var}(Y)$$

$$= \sigma_Z^2 + \sigma_Y^2.$$

$$\text{cor}(U,V) = \frac{\text{cov}(U,V)}{\sqrt{\text{var}(U)\,\text{var}(V)}} = \frac{\sigma_Z^2}{\sqrt{(\sigma_Z^2+\sigma_X^2)(\sigma_Z^2+\sigma_Y^2)}}$$

(b)

$$M_Y(t) = E(e^{tY}) = E(e^{t\beta X})$$

$$= M_X(t\beta)$$

$$= \left(\frac{\lambda}{\lambda + t\beta}\right)^{\alpha}$$

$$= \left(\frac{\lambda/\beta}{\lambda/\beta + t}\right)^{\alpha},$$ the MGF of a $\Gamma(\alpha, \lambda/\beta)$. ∎

Question 3.

X = "annual number of hurricanes" $\sim$ Poisson (λ)

Y = "number of times the hospital is damaged by a hurricane" $\sim$ Bin (X, p).

$$P(Y=0) = \sum_{k=0}^{\infty} P(Y=0 \mid X=k)\, P(X=k)$$

We are conditioning on X using the law of total probability.

$$= \sum_{k=0}^{\infty} (1-p)^k \, e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} e^{\lambda(1-p)} \underbrace{\sum_{k=0}^{\infty} e^{-\lambda(1-p)} \frac{\{(1-p)\lambda\}^k}{k!}}_{=1}$$

$\sim$ Poisson $((1-p)\lambda)$.

$$= e^{-\lambda + \lambda - \lambda p}$$

$$= e^{-\lambda p}$$

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Question 4.

Two solutions were possible : A- Using a bivariate change of variable.

B- Evaluating the cummulative distribution function.

Solution A.

$X \sim \exp(\lambda)$

$Y \sim \exp(\mu)$

$$E(X) = \frac{1}{\lambda} = \frac{1}{\mu} = E(Y) \quad \Rightarrow \quad \lambda = \mu.$$

$$U = \frac{X}{X+Y} \quad \in [0, 1] \qquad X = V$$

$$V = X \quad \in [0, \infty) \qquad Y = \frac{V}{U} - V = \left(\frac{1}{U} - 1\right) V$$

$$|\partial J| = \begin{vmatrix} 0 & \frac{-V}{U^2} \\ 1 & \frac{1}{U} - 1 \end{vmatrix} = \left|\frac{V}{U^2}\right| = \frac{V}{U^2} \quad \text{(it is always positive)}$$

$$f_{UV}(u,v) = f_{XY}\left(v, \left(\tfrac{1}{u}-1\right)v\right) \cdot \frac{v}{u^2}$$

$$= \lambda^2 e^{-\lambda v} e^{-\lambda\left(\frac{1}{u}-1\right)v} \frac{v}{u^2} = \lambda^2 \frac{v}{u^2} e^{-\lambda \frac{v}{u}}$$

$$f_U(u) = \int_0^\infty f_{U,V}(u,v)\, dv$$

$$= \frac{\lambda^2}{u^2} \frac{\Gamma(2)}{(\lambda/u)^2} \int_0^\infty \underbrace{\frac{(\lambda/u)^2}{\Gamma(2)} v\, e^{-\frac{\lambda}{u} v}}_{\sim \Gamma(2, \lambda/u)}\, dv.$$

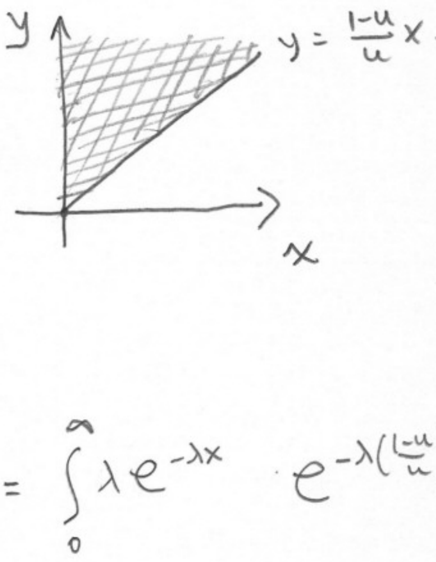
$$= 1 \qquad \text{for } u \in [0,1].$$

Therefore, $U \sim \text{unif}(0,1)$. ∎

<u>Solution B.</u>

$$F_U(u) = P(U \le u) = P\left(\frac{X}{X+Y} \le u\right)$$

$$= P\left(Y \geq \frac{1-u}{u} X\right)$$



$$= \int_0^\infty \int_{\frac{1-u}{u}x}^\infty f_{XY}(x,y)\, dy\, dx.$$

$$= \int_0^\infty \int_{\frac{1-u}{u}x}^\infty \lambda^2 e^{-\lambda x} e^{-\lambda y}\, dy\, dx = \int_0^\infty \lambda e^{-\lambda x} \cdot e^{-\lambda\left(\frac{1-u}{u}\right)x}\, dx$$

$$= \int_0^\infty \lambda e^{-\frac{\lambda}{u}(u+1-u)x} dx = u \underbrace{\int_0^\infty \frac{\lambda}{u} e^{-\frac{\lambda}{u}x}\, dx}_{=1 \text{ since } \sim \exp(\lambda/u)}$$

$= u$, the CDF of a unif(0,1). ∎