MIDTERM EXAMINATION # 2

Statistics 305

Term 1, 2006-2007

Thursday, November 9, 2006

Time: 9:30am – 10:50am

Student Name (**Please print in caps**): SOLUTIONS

Student Number: ______________________

Notes:

- This midterm has 5 problems on the 6 following pages, plus 3 pages of statistical tables. Check to ensure that you have a complete paper.
- The amount each part of each question is worth is shown in [ ] on the left-hand side of the page.
- Where appropriate, record your answers in the blanks provided on the right-hand side of the page.
- Your solutions must be justified; show all the work and state all the reason(s) leading to your answer for each question in the space provided immediately under the question.
- Clear and complete solutions are essential; little partial credit will be given.
- This is a closed book exam.
- A single one-sided 8.5 x 11 page of notes is allowed.
- Calculators are allowed (but not for symbolic differentiation or integration).
- No devices (including calculators) that can store text or send/receive messages are allowed.

Problem	Total Available	Score
1.	7	
2.	7	
3	9	
4.	21	
5.	6	
Total	50	

1. Suppose $X_1, X_2, \ldots, X_n$ is a simple random sample from a normal population with a mean of μ and variance of σ^2, where both parameters are unknown. Derive the forms of the *exact* $1-\alpha$ confidence intervals for:

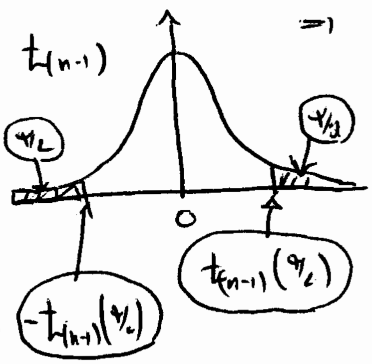
[3] a) the population mean μ.

$$\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t_{(n-1)}$$

$$\Rightarrow 1-\alpha = P_r\left\{-t_{(n-1)}(\alpha/2) \le \frac{\bar{X}-\mu}{S/\sqrt{n}} \le t_{(n-1)}(\alpha/2)\right\}$$

$$= P_r\left\{-t_{(n-1)}(\alpha/2)\cdot S/\sqrt{n} \le \bar{X}-\mu \le t_{(n-1)}(\alpha/2)\cdot S/\sqrt{n}\right\}$$

$$= P_r\left\{\bar{X}-t_{(n-1)}(\alpha/2)S/\sqrt{n} \le \mu \le \bar{X}+t_{(n-1)}(\alpha/2)\cdot S/\sqrt{n}\right\}$$



$$\Rightarrow \left(\bar{X}-t_{(n-1)}(\alpha/2)\cdot S/\sqrt{n},\ \bar{X}+t_{(n-1)}(\alpha/2)\cdot S/\sqrt{n}\right)$$ is an exact $1-\alpha$ CI for μ

[4] b) the population standard deviation σ.

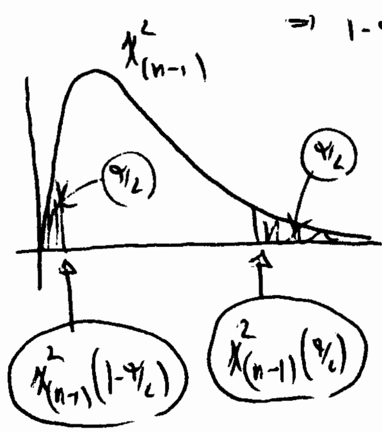
$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}$$

$$\Rightarrow 1-\alpha = P_r\left\{\chi^2_{(n-1)}(1-\alpha/2) \le \frac{(n-1)S^2}{\sigma^2} \le \chi^2_{(n-1)}(\alpha/2)\right\}$$

$$= P_r\left\{\frac{1}{\chi^2_{(n-1)}(1-\alpha/2)} \ge \frac{\sigma^2}{(n-1)S^2} \ge \frac{1}{\chi^2_{(n-1)}(\alpha/2)}\right\}$$

$$= P_r\left\{\frac{(n-1)S^2}{\chi^2_{(n-1)}(1-\alpha/2)} \ge \sigma^2 \ge \frac{(n-1)S^2}{\chi^2_{(n-1)}(\alpha/2)}\right\}$$

$$= P_r\left\{\sqrt{\frac{(n-1)S^2}{\chi^2_{(n-1)}(\alpha/2)}} \le \sigma \le \sqrt{\frac{(n-1)S^2}{\chi^2_{(n-1)}(1-\alpha/2)}}\right\}$$



$$\Rightarrow \left(\sqrt{\frac{(n-1)S^2}{\chi^2_{(n-1)}(\alpha/2)}},\ \sqrt{\frac{(n-1)S^2}{\chi^2_{(n-1)}(1-\alpha/2)}}\right)$$ is an exact $1-\alpha$ CI for σ

2. Suppose a simple random sample of $n = 9$ from a normal population leads to a sample average of $\bar{x} = 22$ and a sample standard deviation of $s = 6$. Evaluate ***exact*** 90% confidence intervals for:

$\boxed{\text{Use results from 1.}}$

[3] a) the population mean μ.

$(18.3, 25.7)$

$$22 \pm t_{(8)}(0.05) \cdot \frac{6}{\sqrt{9}}$$

$$\Leftrightarrow 22 \pm 1.860 \cdot 2$$

$$\Leftrightarrow 22 \pm 3.72 \Rightarrow (18.28, 25.72)$$

[4] b) the population standard deviation σ.

$(4.3, 10.3)$

$$\left(\sqrt{\frac{8}{\chi^2_{(8)}(0.05)}} \cdot 6, \sqrt{\frac{8}{\chi^2_{(8)}(0.95)}} \cdot 6 \right)$$

$$\Leftrightarrow \left(\sqrt{\frac{8}{15.51}} \cdot 6, \sqrt{\frac{8}{2.73}} \cdot 6 \right)$$

$$\Leftrightarrow \left(\sqrt{0.5158} \cdot 6, \sqrt{2.9304} \cdot 6 \right)$$

$$\Leftrightarrow (4.3091, 10.2711)$$

3. Suppose $X_1, X_2, \ldots, X_n$ is a simple random sample from the distribution:

$$f_\theta(x) = \theta x^{\theta - 1} \qquad \text{for } 0 \le x \le 1.$$

Note that this is a density function provided that $\theta > 0$.

[3] a) Evaluate the Fisher Information in a single observation X. $1/\theta^2$

$$\log f = \log\theta + (\theta-1)\log x$$

$$\frac{d}{d\theta}\log f = \frac{1}{\theta} + \log x \qquad I(\theta)$$

$$\frac{d^2}{d\theta^2}\log f = -\frac{1}{\theta^2} \Rightarrow \underbrace{-E\left(\frac{d^2}{d\theta^2}\log f\right)}_{=I(\theta)} = \frac{1}{\theta^2}$$

Note: Using the definition $I(\theta) = E\left[\left(\frac{1}{\theta} + \log X\right)^2\right]$ leads to the same answer but the calculation is much harder!

[3] b) Find $\hat{\theta}_{ML}$, the maximum likelihood estimator (MLE) of θ. $\frac{1}{-\overline{\log X}}$

From (a) $\frac{d}{d\theta}\log f(x_i|\theta) = \frac{1}{\theta} + \log x_i$

$\boxed{\text{Note: } 0<x<1 \Rightarrow \log x < 0 \Rightarrow -\log x > 0}$

$$\Rightarrow \frac{d}{d\theta}\ell(\theta) = \frac{n}{\theta} + \sum_{i=1}^n \log x_i$$

$$\Rightarrow \ell'(\theta) = 0 \iff \theta = \frac{1}{-\frac{1}{n}\sum_{i=1}^n \log x_i} = \frac{1}{-\overline{\log x}}$$

* Check that it is a max:

$$\ell'(\theta) > 0 \iff \theta < \frac{1}{-\overline{\log x}} \Rightarrow \text{it is a max} \Rightarrow \hat{\theta}_{ML} = \frac{1}{-\overline{\log X}}$$

$\boxed{\text{Alternatively, } \ell''(\theta) = -\frac{n}{\theta^2} \Rightarrow \text{it is a max}}$

[3] c) Derive the form of the approximate $1-\alpha$ confidence interval for θ based on the MLE $\hat{\theta}_{ML}$.

$$\hat{\theta}_{ML} \approx N\left(\theta, \frac{1}{nI(\theta)}\right) \Rightarrow \hat{\theta}_{ML} \pm z_{\alpha/2}\sqrt{\frac{1}{nI(\theta)}} \text{ is an approximate } 1-\alpha \text{ CI for } \theta$$

[basically from 1.(a)]

$$nI(\theta) = E\left[-\ell''(\theta)\right] = E\left(\frac{n}{\theta^2}\right) = \frac{n}{\theta^2} \text{ from (b)}$$

$$\Rightarrow \hat{\theta}_{ML} \pm z_{\alpha/2}\cdot \frac{\theta}{\sqrt{n}} \text{ is an approximate } 1-\alpha \text{ CI for } \theta$$

But θ is unknown $\Rightarrow \hat{\theta}_{ML} \pm z_{\alpha/2}\cdot\frac{\hat{\theta}_{ML}}{\sqrt{n}}$ is the desired approximate $1-\alpha$ CI for θ

(OK because $\hat{\theta}_{ML} \xrightarrow{P} \theta$)

4. Suppose $X_1, X_2, \ldots, X_n$ is a simple random sample from the Rayleigh distribution:

$$f_\theta(x) = (x/\theta^2)\exp(-x^2/2\theta^2) \qquad \text{for } x \geq 0,$$

where $\theta > 0$.

[4] a) Find $\hat{\theta}_{MM}$, the method of moments estimator (MME) of θ. $\sqrt{\frac{2}{\pi}}\,\bar{X}$

$$\mu_1 = E(X) = \int_0^\infty x\left(\frac{x}{\theta^2}\right)\exp\left(-\frac{x^2}{2\theta^2}\right)dx$$

By symmetry

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{\theta^2}\exp\left(-\frac{x^2}{2\theta^2}\right)dx$$

$$= \frac{1}{2}\sqrt{2\pi}\,\frac{1}{\theta}\int_{-\infty}^{\infty}\frac{x^2}{\sqrt{2\pi}\,\theta}\exp\left(-\frac{x^2}{2\theta^2}\right)dx$$

$$= \sqrt{\frac{\pi}{2}}\,\frac{1}{\theta}\,E(X^2) \quad \text{where } X \sim N(0,\theta^2)$$

$$= \sqrt{\frac{\pi}{2}}\,\frac{1}{\theta}\cdot\theta^2 = \sqrt{\frac{\pi}{2}}\,\theta$$

$$\theta = \sqrt{\frac{2}{\pi}}\,\mu_1$$

$$\Rightarrow \hat{\theta}_{MM} = \sqrt{\frac{2}{\pi}}\,\hat{\mu}_1 = \sqrt{\frac{2}{\pi}}\,\bar{X}$$

Alternatively, transform:

$$v = \frac{x^2}{2\theta^2} \Rightarrow dv = \frac{x}{\theta^2}dx$$

$$\Rightarrow \mu_1 = \int_0^\infty (2\theta^2 v)^{1/2}\exp(-v)\,dv$$

$$= \sqrt{2}\,\theta\int_0^\infty v^{3/2-1}\exp(-v)\,dv$$

$$= \sqrt{2}\,\theta\,\Gamma(3/2)$$

$$= \sqrt{2}\,\theta\,\frac{1}{2}\Gamma(1/2)$$

$$= \frac{\theta}{\sqrt{2}}\sqrt{\pi}$$

[5] b) Find the exact variance of $\hat{\theta}_{MM}$, the MME of θ. $\frac{(4-\pi)}{\pi}\cdot\frac{\theta^2}{n}$

$$Var(\hat{\theta}_{MM}) = \frac{2}{\pi}Var(\bar{X})$$

$$= \frac{2}{\pi}\,\frac{1}{n}\,Var(X)$$

$$\Rightarrow \text{Need to evaluate } E(X^2) = \int_0^\infty x^2\left(\frac{x}{\theta^2}\right)\exp\left(-\frac{x^2}{2\theta^2}\right)dx$$

$$\text{Transform as above} = \int_0^\infty (2\theta^2 v)\exp(-v)\,dv$$

$$= 2\theta^2\int_0^\infty v^{2-1}\exp(-v)\,dv$$

$$= 2\theta^2\,\Gamma(2) = 2\theta^2$$

$$\Rightarrow Var(X) = 2\theta^2 - \left(\sqrt{\pi/2}\,\theta\right)^2 = \theta^2\left(2 - \pi/2\right) = \frac{(4-\pi)}{2}\theta^2$$

$$\Rightarrow Var(\hat{\theta}_{MM}) = \frac{(4-\pi)}{\pi}\,\frac{\theta^2}{n}$$

$$\sqrt{\tfrac{1}{2}\overline{X^2}}$$

[4] c) Find $\hat{\theta}_{ML}$, the maximum likelihood estimator of θ.

$$\log f = \log x - 2\log\theta - x^2/2\theta^2$$

$$\Rightarrow \ell(\theta) = \sum_{i=1}^{n}\log x_i - 2n\log\theta - \frac{1}{2\theta^2}\sum_{i=1}^{n}x_i^2$$

$$\Rightarrow \ell'(\theta) = -\frac{2n}{\theta} - \frac{(-2)}{2\theta^3}\, n\overline{x^2} = -\frac{2n}{\theta} + \frac{n\overline{x^2}}{\theta^3}$$

$$\Rightarrow \ell'(\theta) = 0 \iff \theta^2 = \frac{1}{2}\overline{x^2} \iff \theta = \sqrt{\tfrac{1}{2}\overline{x^2}}$$

* Check it is a max: $\ell'(\theta) > 0 \iff \theta^2 < \frac{1}{2}\overline{x^2} \iff 0 < \theta < \sqrt{\frac{1}{2}\overline{x^2}} \Rightarrow$ it is a max!

$$\Rightarrow \hat{\theta}_{ML} = \sqrt{\tfrac{1}{2}\overline{X^2}}$$

Alternatively $\ell''(\theta) = \frac{2n}{\theta^2} - \frac{3n\overline{x^2}}{\theta^4} \Rightarrow \ell''\left(\sqrt{\tfrac{1}{2}\overline{x^2}}\right) = \frac{2n}{(\frac{1}{2}\overline{x^2})} - \frac{3n\overline{x^2}}{(\frac{1}{2}\overline{x^2})^2}$

$$= \frac{4n}{\overline{x^2}} - \frac{12n\overline{x^2}}{(\overline{x^2})^2} = \frac{-8n}{\overline{x^2}} < 0$$

$\Rightarrow$ it is a max!

[4] d) Find the asymptotic variance of $\hat{\theta}_{ML}$, the MLE of θ.

$$A.Var(\hat{\theta}_{ML}) = \frac{1}{nI(\theta)}$$

But $$nI(\theta) = -E\left[\frac{2n}{\theta^2} - \frac{3}{\theta^4}\sum_{i=1}^{n}X_i^2\right]$$

$$= -\frac{2n}{\theta^2} + \frac{3}{\theta^4}\, n\, E(X^2)$$

But $E(X^2) = 2\theta^2$ from (b) $$= -\frac{2n}{\theta^2} + \frac{3n}{\theta^4}\, 2\theta^2 = \frac{4n}{\theta^2}$$

$$\Rightarrow A.Var(\hat{\theta}_{ML}) = \frac{\theta^2}{4n}$$

[4] e) Evaluate the asymptotic relative efficiency of the MME relative to the MLE. What is the practical interpretation of this result?

$$ARE(\hat{\theta}_{MM}, \hat{\theta}_{ML}) = \frac{A.Var(\hat{\theta}_{ML})}{Var(\hat{\theta}_{MM})}$$

$$= \frac{\theta^2/4n}{\frac{(4-\pi)}{\pi}\cdot\theta^2/n} = \frac{\pi}{4(4-\pi)} \doteq 0.915$$

because $Bias(\hat{\theta}_{ML}) = \frac{1}{n}h(\theta) + \cdots$

$\Rightarrow Bias^2(\hat{\theta}_{ML}) \sim \frac{1}{n^2}h^2(\theta)$

So $Bias^2(\hat{\theta}_{ML})$ can be ignored in $MSE(\hat{\theta}_{ML})$ because it is of smaller order than $A.Var(\hat{\theta}_{ML})$

$\Rightarrow$ MME requires about 9% larger sample size to achieve same variance as MLE

5. Suppose $X_1, X_2, \ldots, X_n$ is a simple random sample from a normal population with mean μ and (known) variance = 1.

[4] a) Show that $\overline{X}$ is a sufficient statistic for μ.

$$f_{\underset{\sim}{X}}(x_1, x_2, \ldots, x_n | \mu) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left\{-\frac{1}{2}\Sigma_i (x_i - \mu)^2\right\}$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left\{-\frac{1}{2}\left[\Sigma_i (x_i - \bar{x})^2 + n(\bar{x}-\mu)^2\right]\right\}$$

$$= \underbrace{\left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left\{-\frac{1}{2}\Sigma_i (x_i - \bar{x})^2\right\}}_{\text{Depends only on the } x_i\text{'s, not on } \mu} \cdot \underbrace{\exp\left\{-\frac{n}{2}(\bar{x}-\mu)^2\right\}}_{\text{Depends on the } x_i\text{'s only through } \bar{x}}$$

$\Rightarrow$ Factorization Theorem yields that $\overline{X}$ is sufficient statistic for μ in this case

[2] b) What is the practical interpretation of this result?

$\overline{X}$ contains all the information about μ in the entire sample $X_1, X_2, \ldots, X_n$ $\Rightarrow$ for purposes of inference, don't need to keep the individual X_i's, only need to keep $\overline{X}$. (Of course, this depends on the normal model being correct. Would need the individual X_i's to check on the adequacy of the model.)

Can then work with the distribution of $\overline{X}$, rather than the joint distribution of $X_1, X_2, \ldots, X_n$.