Midterm II

9 November 2004, 12:30 – 13:50
Time allowed : 80 minutes.

Authorized material :

- One letter-size cheat sheet (2-sided).
- One scientific calculator without wireless communication feature.

Instructions :

- The exam has 6 pages including this one.
- Note that **formulae** are provided in **page 6**, read them now. You can detach that page if convenient; if you do so, do not write answers on it.
- Answer all 4 questions; the total number of points is 100.
- Write legibly; give complete solutions.
- You can use the back side of the sheets as drafts. If you use it for writing answers, indicate it clearly.

Last Name : ______________________________

First Name : ______________________________

Student Number : ______________________________

Signature : ______________________________

Question 1

The sample $X_1, \ldots, X_n$ comes from a log-normal distribution with σ^2 known and μ unknown.

a) [10 pts] Consider $\tilde{\mu}$, an unbiased estimator of μ. What is the minimal variance that $\tilde{\mu}$ could possibly achieve.

b) [10 pts] Calculate the MSE of $\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} \log(X_i)$, the MLE of μ.

Question 2

A Bayesian analysis is performed on a sample $X_1, \ldots, X_n$ from a Weibull distribution (see page 6 to use the right parameterization). The parameter α is known and the prior knowledge on λ is represented by a gamma distribution with parameters a and b.

a) [15 pts] What is the posterior distribution of λ?

b) [5 pts] What estimate of p will you use if your loss function is $\ell(p, \hat{p}) = (p - \hat{p})^2$?

c) [10 pts] The shape of a log-normal distribution is similar to that of a gamma distribution. If the the log-normal distribution is used to model the prior knowledge on λ, the posterior distribution has a complicated form which is not from a known family. Explain what strategy you would use to answer question b in that case? Give some details without being technical (name the tools you use, but do not write any formula).

Question 3

[20 pts] The sample $X_1, \ldots, X_n$ follows an Inverse-Gaussian distribution. Find the MLE of λ when μ is known.

Question 4

Consider a sample $X_1, \ldots, X_n$ from a log-normal distribution with μ and σ^2 both unknown.

a) [20 pts] Use the method of moments to find estimators for μ and σ^2.

b) [10 pts] Find a sufficient statistic of low dimension for the log-normal with both parameters unknown. *[Hint: Expand the square.]*

Useful formulae

Inverse-Gaussian distribution with parameters $\mu > 0$ and $\lambda > 0$.

$$
\begin{aligned}
f(x) &= \begin{cases} \left(\frac{\lambda}{2\pi x^3}\right)^{\frac{1}{2}} e^{-\frac{\lambda}{2\mu^2 x}(x-\mu)^2} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} \\
\mathrm{E}(X) &= \mu \\
\mathrm{var}(X) &= (\mu\lambda)^2
\end{aligned}
$$

Log-Normal distribution with parameters μ and $\sigma^2 > 0$

$$
\begin{aligned}
f(x) &= \begin{cases} \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma x} e^{-\frac{\{\log(x)-\mu\}^2}{2\sigma^2}} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases} \\
\mathrm{E}(X) &= e^{\mu+\sigma^2/2} \\
\mathrm{var}(X) &= e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)
\end{aligned}
$$

Note that when X follows a log-normal with parameters μ and σ^2, then $\log(X) \sim N(\mu, \sigma^2)$.

Weibull distribution with parameters $\alpha > 0$ and $\lambda > 0$.

$$
\begin{aligned}
f(x) &= \begin{cases} \alpha\lambda x^{\alpha-1} e^{-\lambda x^{\alpha}} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases} \\
\mathrm{E}(X) &= \frac{\Gamma(1 + 1/\alpha)}{\lambda^{1/\alpha}} \\
\mathrm{var}(X) &= \frac{\Gamma(1 + 2/\alpha) - \Gamma(1 + 1/\alpha)^2}{\lambda^{2/\alpha}}
\end{aligned}
$$