QUIZ # 2

Statistics 305

Term 2, 2006-2007

Tuesday, February 6, 2007

Time: 2:00pm – 2:30pm

Student Name (**Please print in caps**): SOLUTIONS

Student Number: ______________________

Notes:

- This quiz has 3 problems on the 4 following pages, plus 1 page of statistical tables. Check to ensure that you have a complete paper.
- The amount each part of each question is worth is shown in [ ] on the left-hand side of the page.
- Where appropriate, record your answers in the blanks provided on the right-hand side of the page.
- Your **solutions must be justified**; show all the work and state all the reasons leading to your answer for each question in the space provided immediately under the question.
- Clear and complete solutions are important; little partial credit will be given.
- This is a closed book exam.
- A **single one-sided 8.5 x 11 page** of notes is allowed.
- Calculators are allowed (but not for symbolic differentiation or integration).
- No devices (including calculators) that can store text or send/receive messages are allowed.

Problem	Total Available	Score
1.	5	
2.	11	
3.	9	
Total	25	

1. Suppose a measurement has a mean of μ and a standard deviation of $\sigma = 3$. We will estimate the value of μ by $\overline{X}$, the average of n such independent measurements. How large a value of n is required to be 80% confident that this estimate will be within 0.1 of the true value; that is, that $P(|\overline{X} - \mu| \le 0.1) = 0.80$? 1,475

[5]

$$P(|\overline{X}-\mu| \le 0.1) = P\left(\frac{|\overline{X}-\mu|}{\sigma/\sqrt{n}} \le \frac{0.1}{\sigma/\sqrt{n}} \right)$$

$$\cong P\left(|Z| \le \frac{\sqrt{n}(0.1)}{\sigma}\right) \text{ by CLT}$$

$$\text{So } \ge 0.80 \iff \frac{\sqrt{n}(0.1)}{\sigma} \ge 1.28$$

provided n is "large enough" to apply CLT

$$\iff \sqrt{n} \ge \frac{1.28\sigma}{0.1} = 12.8\sigma$$

$$\text{But } \sigma = 3 \iff \sqrt{n} \ge 38.4$$

$$\iff n \ge 1{,}474.56$$

$$\iff \underline{\underline{n \ge 1{,}475}}$$

(and this value of n is certainly "large enough" to apply CLT in most contexts, so all is OK!)

2. Suppose X is a positive random variable with a mean of μ_X and a variance of σ_X^2. If $Y = -\ln(X)$, where $\ln \equiv \log_e$, use the delta method to obtain expressions (entirely in terms of μ_X and σ_X^2) for:

Note: You do <u>not</u> need to derive the basic results for the delta method as done in class but you must state clearly the exact form of the results you are using.

[2] a) a "first-order" approximation to $E(Y)$. $-\ln \mu_X$

$Y = g(X)$, where $g(x) = -\ln x$

Delta method $\Rightarrow$ $E(Y) \cong g(E(X)) = -\ln \mu_X$

[4] b) a "first-order" approximation to $SD(Y)$. σ_X / μ_X

Delta method $\Rightarrow$ $\mathrm{Var}(Y) \cong [g'(E(X))]^2 \cdot \mathrm{Var}(X)$

$g(x) = -\ln x$

$\Rightarrow g'(x) = -\frac{1}{x}$

$$= \left(-\frac{1}{\mu_X}\right)^2 \cdot \sigma_X^2$$

$$= \frac{\sigma_X^2}{\mu_X^2}$$

$$\Rightarrow SD(Y) \cong \frac{\sigma_X}{\mu_X}$$

[3] c) a “second-order” approximation to $E(Y)$.

$$-\ln\mu_X + \frac{\sigma_X^2}{2\mu_X^2}$$

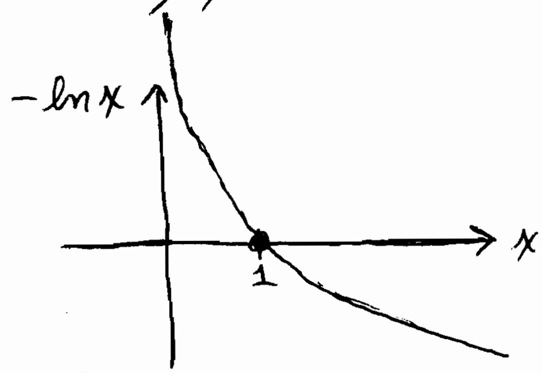
Delta method $\Rightarrow$ $E(Y) \triangleq g(E(X)) + \frac{1}{2} g''(E(X)) \cdot Var(X)$

$g'(x) = -\frac{1}{x}$

$\Rightarrow g''(x) = \frac{1}{x^2}$

$$= -\ln\mu_X + \frac{1}{2}\left(\frac{1}{\mu_X}\right)^2 \cdot \sigma_X^2$$

$$= -\ln\mu_X + \frac{\sigma_X^2}{2\mu_X^2}$$



[2] d) If X was uniformly distributed on [0,1], do you expect the delta method approximations to the moments of $Y = -\ln(X)$ to be fairly accurate? Explain.

Do not expect approximations to be very accurate because:

① $-\ln x$ changes rapidly (is highly non-linear) on $[0,1]$
[$g'(x) = -\frac{1}{x}$ which is far from constant on $[0,1]$]

② the distribution of X is not highly concentrated around the mean μ_X; that is, σ_X^2 is rather large (for distributions which are positive only on $[0,1]$).

Remarks: In fact, if $X \sim U[0,1]$, then $Y = -\ln X \sim \exp(1) \equiv \mathcal{G}(1,1)$

$\Rightarrow E(Y) = Var(Y) = 1$ whereas: (a) $= -\ln(\frac{1}{2}) = \ln(2) \cong 0.693$

(b) $= \frac{1}{\sqrt{12}} / \frac{1}{2} = \frac{1}{\sqrt{3}} \cong 0.577$

(c) $= \ln(2) + \frac{1}{6} \cong 0.860$

$\Rightarrow$ These approximations are not very good!

3. Suppose X and Y are bivariate normally distributed random variables with means μ_X, μ_Y, variances σ_X^2, σ_Y^2, and correlation ρ_{XY}. Then $M_{X,Y}(s,t)$, the joint moment generating function of X and Y evaluated at s and t, is given by:

$$M_{X,Y}(s,t) = \exp(s\mu_X + t\mu_Y + [s^2\sigma_X^2 + 2st\rho_{XY}\sigma_X\sigma_Y + t^2\sigma_Y^2]/2).$$

Let $W = aX + bY + c$, where a, b and c are constants.

[6] a) Evaluate $M_W(t)$, the moment generating function of W evaluated at t. ________________

$$M_W(t) = E(e^{tW})$$

$$= E\left(e^{t[aX+bY+c]}\right) = e^{tc} E\left(e^{(at)X+(bt)Y}\right) = e^{tc} M_{X,Y}(at, bt)$$

$$= e^{tc} \exp\left\{(at)\mu_X + (bt)\mu_Y + \left[(at)^2\sigma_X^2 + 2(at)(bt)\rho_{XY}\sigma_X\sigma_Y + (bt)^2\sigma_Y^2\right]/2\right\}$$

$$= e^{tc} \exp\left\{(a\mu_X + b\mu_Y)t + \frac{t^2}{2}\left[\underbrace{a^2\sigma_X^2 + 2ab\rho_{XY}\sigma_X\sigma_Y + b^2\sigma_Y^2}\right]\right\}$$

$$= \exp\left\{[a\mu_X + b\mu_Y + c]t + [\quad] t^2/2\right\}, \text{ for } \underline{\underline{\text{any}}} \text{ value of } t$$

[3] b) What is the distribution of W? ________________

Note: Explain clearly how this result follows from a).

If $V \sim N(\mu, \sigma^2)$, then $M_V(t) = \exp\{\mu t + \sigma^2 t^2/2\}$

$M_W(t)$ has exactly this form, with $\mu = a\mu_X + b\mu_Y + c$

and $\sigma^2 = a^2\sigma_X^2 + 2ab\rho_{XY}\sigma_X\sigma_Y + b^2\sigma_Y^2$

$\Rightarrow$ By the Uniqueness Theorem, we have the result that $W \sim N(\mu, \sigma^2)$, where μ and σ^2 are given by the above expressions.

THE END