QUIZ # 5

Statistics 305

Term 2, 2006-2007

Thursday, March 29, 2007

Time: 2:00pm – 3:00pm

Student Name (**Please print in caps**): SOLUTIONS

Student Number: ____________________

Notes:

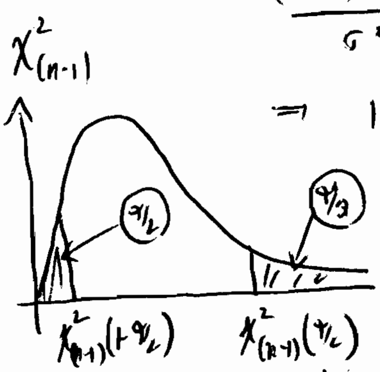
- This quiz has 3 problems on the 5 following pages, plus 3 pages of statistical tables. Check to ensure that you have a complete paper.
- The amount each part of each question is worth is shown in [ ] on the left-hand side of the page.
- Where appropriate, record your answers in the blanks provided on the right-hand side of the page.
- Your solutions **must be justified**; show **all the work** and state **all the reasons** leading to your answer for each question in the space provided immediately under the question.
- Clear and complete solutions are essential; little partial credit will be given.
- This is a closed book exam.
- A single one-sided 8.5 x 11 page of notes is allowed.
- Calculators are allowed (but not for symbolic differentiation or integration).
- No devices (including calculators) that can store text or send/receive messages are allowed.

Problem	Total Available	Score
1.	7	
2.	9	
3.	9	
Total	25	

1. Suppose $X_1, X_2, \ldots, X_n$ is a simple random sample from a normal population with a mean of μ and variance of σ^2, where both parameters are unknown.

[4] a) Derive the form of the **exact** $1-\alpha$ confidence interval for the population standard deviation σ.

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}$$



$$\Rightarrow \quad 1-\alpha = \Pr\left\{\chi^2_{(n-1)}(1-\alpha/2) \le \frac{(n-1)S^2}{\sigma^2} \le \chi^2_{(n-1)}(\alpha/2)\right\}$$

$$= \Pr\left\{\frac{1}{\chi^2_{(n-1)}(1-\alpha/2)} \geq \frac{\sigma^2}{(n-1)S^2} \geq \frac{1}{\chi^2_{(n-1)}(\alpha/2)}\right\}$$

Using the same notation as in class

$$= \Pr\left\{\frac{(n-1)S^2}{\chi^2_{(n-1)}(1-\alpha/2)} \geq \sigma^2 \geq \frac{(n-1)S^2}{\chi^2_{(n-1)}(\alpha/2)}\right\}$$

$$\Rightarrow \left(\frac{(n-1)S^2}{\chi^2_{(n-1)}(\alpha/2)}, \frac{(n-1)S^2}{\chi^2_{(n-1)}(1-\alpha/2)}\right) \text{ is \underline{exact} } 1-\alpha \text{ CI for } \sigma^2$$

$$\Rightarrow \left(\sqrt{\frac{(n-1)}{\chi^2_{(n-1)}(\alpha/2)}}\,S, \sqrt{\frac{(n-1)}{\chi^2_{(n-1)}(1-\alpha/2)}}\right) \text{ is \underline{exact} } 1-\alpha \text{ CI for } \underline{\underline{\sigma}}$$

[3] b) Suppose a simple random sample of $n = 5$ from this population leads to a sample average of $\bar{x} = 22.1$ and a sample standard deviation of $s = 3.7$. Evaluate the **exact** 90% confidence interval for the population standard deviation σ. $(2.4, 8.0)$

$n=5 \Rightarrow n-1 = 4 \Rightarrow \chi^2_{(4)}$ is the relevant distribution

90% CI $\Rightarrow$ Need $\chi^2_{(4)}(0.95) = 0.711$

and $\chi^2_{(4)}(0.05) = 9.49$

$$\Rightarrow \left(\sqrt{\frac{4}{9.49}} \times 3.7, \sqrt{\frac{4}{0.711}} \times 3.7\right)$$

$$= (2.402, 8.776) \simeq (2.4, 8.8)$$

2. Suppose $X_1, X_2, \ldots, X_n$ is a simple random sample from a population having an exponential distribution with mean $\theta\ (\theta > 0)$; that is, having the density function:

$$f_\theta(x) = \frac{1}{\theta}\exp\left(-\frac{x}{\theta}\right) \qquad \text{for } x > 0.$$

[3] a) Find the expression for $\hat{\theta}_{ML}$, the maximum likelihood estimator (MLE) of θ. $\overline{X}$

$$L(\theta) = \prod_{i=1}^{n} f_\theta(x_i) = \frac{1}{\theta^n}\exp\left(-\frac{\sum_{i=1}^{n} x_i}{\theta}\right) = \frac{1}{\theta^n}\exp\left(-\frac{n\bar{x}}{\theta}\right)$$

Note: $L(\theta) > 0$ for $0 < \theta < \infty$ and $L(\theta) \to 0$ as $\theta \to 0$
and as $\theta \to \infty$

So max will be achieved somewhere inside $0 < \theta < \infty$
(not on boundaries)

$$\Rightarrow \ell(\theta) = \log L(\theta) = -n\log\theta - n\bar{x}/\theta$$

$$\ell'(\theta) = -\frac{n}{\theta} + \frac{n\bar{x}}{\theta^2}$$

So $\ell'(\theta) > 0 \Leftrightarrow \frac{n\bar{x}}{\theta^2} > \frac{n}{\theta}$
$\Leftrightarrow \theta < \bar{x}$

$\Rightarrow \ell(\theta)$ is increasing for $\theta < \bar{x}$, and decreasing for $\theta > \bar{x}$
$\Rightarrow$ max at $\bar{x} \Rightarrow \hat{\theta}_{ML} = \overline{X}$

Alternately, $\ell''(\theta) = \frac{n}{\theta^2} - \frac{2n\bar{x}}{\theta^3}$

When $\theta = \bar{x} \Rightarrow \ell''(\bar{x}) = \frac{n}{\bar{x}^2} - \frac{2n\bar{x}}{\bar{x}^3} = \frac{n}{\bar{x}^2} - \frac{2n}{\bar{x}^2}$

$\Rightarrow \ell''(\bar{x}) = -\frac{n}{\bar{x}^2} < 0 \Rightarrow$ max

[3] b) Show that the MLE $\hat{\theta}_{ML}$ is unbiased and evaluate its **exact** variance.

$\hat{\theta}_{ML} = \overline{X} \Rightarrow E(\hat{\theta}_{ML}) = E(\overline{X}) = E(X) = \theta$

$E(X) = \theta$ is given in the question

$\Rightarrow \hat{\theta}_{ML}$ is unbiased

$$Var(\hat{\theta}_{ML}) = Var(\overline{X}) = \frac{1}{n}Var(X)$$

But $X \sim \mathcal{G}(1, \frac{1}{\theta}) \Rightarrow E(X) = \theta$ (as given in the question)

$\Rightarrow Var(X) = \theta^2$

$$\Rightarrow Var(\hat{\theta}_{ML}) = \frac{1}{n}\theta^2$$

EXACT!

[3] c) Is there any unbiased estimator of the parameter θ that has smaller variance than the MLE $\hat{\theta}_{ML}$? Be sure to explain your reasoning clearly.

No

Cramer-Rao lower bound says that if T is any unbiased estimator for θ, then $Var(T) \geq \frac{1}{nI(\theta)}$

$\Rightarrow$ If $\frac{1}{nI(\theta)} = \frac{\theta^2}{n}$, then $\hat{\theta}_{ML}$ achieves the CRLB and we would have the result desired $\Rightarrow$ No unbiased estimator could have a smaller variance.

$\Rightarrow$ Need to evaluate CRLB:

But $nI(\theta) = E\left[(l'(\theta))^2\right] = E\left[-l''(\theta)\right]$

From (a), we have $-l''(\theta) = -\frac{n}{\theta^2} + \frac{2n\bar{X}}{\theta^3}$

$$\Rightarrow E\left(-l''(\theta)\right) = -\frac{n}{\theta^2} + \frac{2n}{\theta^3}E(\bar{X})$$

$$= -\frac{n}{\theta^2} + \frac{2n}{\theta^3}\theta \quad \text{(from (b))}$$

$$= -\frac{n}{\theta^2} + \frac{2n}{\theta^2}$$

$$= \frac{n}{\theta^2}$$

$\Rightarrow$ CRLB $= \frac{1}{nI(\theta)} = \frac{\theta^2}{n}$ and we have the desired result!

3. Suppose $X_1, X_2, \ldots, X_n$ is a random sample of size n from a $N(\mu, 1)$ population.

[4] a) Use the Neyman-Pearson Lemma to find the form of the most powerful test for testing $H_0: \mu = 0$ versus $H_1: \mu = 1$.

Reject H_0 if $\overline{X} > k$

Neyman-Pearson Lemma tells us that the most powerful test for a simple H_0 versus a simple H_1 is obtained by:

Accept $H_0 \Leftrightarrow \frac{L_0}{L_1} \geq c$ (OR) Reject $H_0 \Leftrightarrow \frac{L_0}{L_1} < c$

$$\text{Now } \frac{L_0}{L_1} = \frac{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}x_i^2\right\}}{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(x_i-1)^2\right\}} = \prod_{i=1}^{n} \exp\left\{-\frac{1}{2}\left[x_i^2 - (x_i-1)^2\right]\right\}$$

$$= \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\left[x_i^2 - (x_i^2 - 2x_i + 1)\right]\right\}$$

$$= \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\left[2x_i - 1\right]\right\}$$

$$= \exp\left\{-\sum_{i=1}^{n}(x_i - \tfrac{1}{2})\right\}$$

$$= \exp\left\{-n(\bar{x} - \tfrac{1}{2})\right\}$$

$$\text{So } \frac{L_0}{L_1} < c \Leftrightarrow \exp\left\{-n(\bar{x} - \tfrac{1}{2})\right\} < c$$

$$\Leftrightarrow -n(\bar{x} - \tfrac{1}{2}) < c' = \log c$$

$$\Leftrightarrow \bar{x} - \tfrac{1}{2} > c'' = -c'/n$$

$$\Leftrightarrow \bar{x} > c'' + \tfrac{1}{2}$$

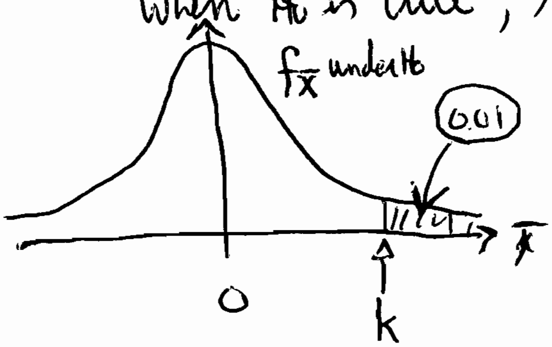
$\Rightarrow$ the form of the MP test is: Reject H_0 if $\overline{X} > k$ (as, of course, you knew had to be the answer!)

[2] b) Write down the explicit form of this test giving the critical value required to achieve a significance level of $\alpha = 0.01$.

Reject H_0 if $\overline{X} > \frac{2.33}{\sqrt{n}}$

Want $0.01 = P_{\mu=0}(\overline{X} > k) \equiv P_{H_0}(\text{Reject } H_0)$

When H_0 is true, $\overline{X} \sim N(0, \frac{1}{n})$ exactly



$\Rightarrow$ Require $k = \frac{z(0.01)}{\sqrt{n}} = \frac{2.33}{\sqrt{n}}$

Algebraically: $P_{\mu=0}(\overline{X} > k) = P_{\mu=0}\left(\frac{\overline{X}-0}{1/\sqrt{n}} > \frac{k-0}{1/\sqrt{n}}\right)$

$= P(Z > \sqrt{n}\,k)$

So, this $= 0.01 \Leftrightarrow \sqrt{n}\,k = z(0.01) = 2.33$

$\Leftrightarrow k = 2.33/\sqrt{n}$

[3] c) Suppose we use this same test for testing $H_0: \mu \le 0$ versus $H_1: \mu > 0$. Evaluate the power function of this test and show it is monotonically increasing in μ. Sketch the power function as a function of μ (roughly). ______________

$$\pi(\mu) \equiv P_\mu(\text{Reject } H_0) = P_\mu\left(\bar{X} > \frac{2.33}{\sqrt{n}}\right)$$

power function

$$= P_\mu\left(\frac{\bar{X}-\mu}{1/\sqrt{n}} > \left(\frac{2.33}{\sqrt{n}} - \mu\right)\Big/\tfrac{1}{\sqrt{n}}\right)$$

$$= P\left(Z > 2.33 - \sqrt{n}\,\mu\right) \quad (1)$$

Any of these is OK

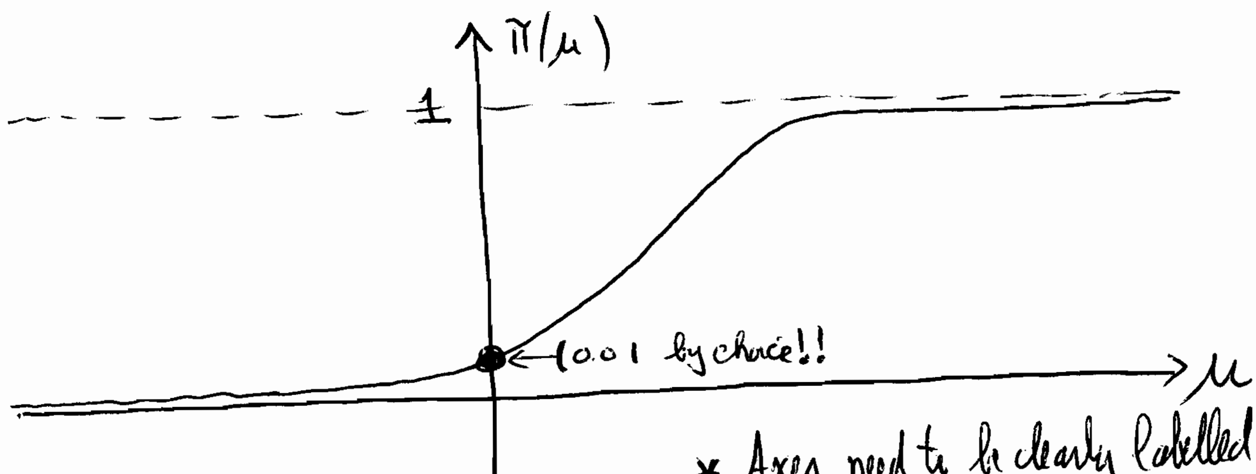
$$= 1 - \Phi\left(2.33 - \sqrt{n}\,\mu\right) \quad (2)$$

$$= \Phi\left(\sqrt{n}\,\mu - 2.33\right) \quad (3)$$

As $\mu \uparrow$: $2.33 - \sqrt{n}\,\mu$ in (1) $\downarrow$, so $\pi(\mu) \uparrow$

ditto in (2)

$\sqrt{n}\,\mu - 2.33$ in (3) $\uparrow$, so $\pi(\mu) \uparrow$



$\mu = 0$

* Axes need to be clearly labelled and $\pi(0) = 0.01$ should be clearly indicated.

THE END