## Lab Assignment # 3: Bootstrapping and Maximum Likelihood

**Question 1.**

*Answer:*

```
> x <- rgamma(5000, shape = 2, rate = 2)
```

We have the following nonlinear equation for $\hat{\alpha}$:

$$n \log \hat{\alpha} - n \log \bar{X} + \sum_{i=1}^{n} \log X_i - n \frac{\Gamma'(\hat{\alpha})}{\Gamma(\hat{\alpha})} = 0 \tag{1}$$

Taking the first derivative (with respect to $\hat{\alpha}$) gives

$$\frac{n}{\hat{\alpha}} - n \left( \frac{\Gamma''(\hat{\alpha})}{\Gamma(\hat{\alpha})} + \frac{1}{\Gamma'(\hat{\alpha})} \right) \tag{2}$$

There is a function, called the psi-gamma function, which is defined by

$$\psi(\alpha) = \log \Gamma(\alpha)$$

Taking the first and second derivatives of this equation yields

$$\frac{d}{d\alpha}\psi(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}$$
$$\frac{d^2}{d\alpha^2}\psi(\alpha) = \frac{\Gamma''(\alpha)}{\Gamma(a)} - \frac{1}{\Gamma'(\alpha)}$$

and these are called the digamma and trigamma functions respectively. Therefore, rewriting equations (1) and (2),

$$n \log \hat{\alpha} - n \log \bar{X} + \sum_{i=1}^{n} \log X_i - n \frac{d}{d\alpha}\psi(\alpha) = 0$$

and

$$\frac{n}{\hat{\alpha}} - n \frac{d^2}{d\alpha^2}\psi(\alpha)$$

Once we have an estimate for $\hat{\alpha}$, then $\hat{\lambda} = \hat{\alpha}/\bar{X}$. We use as an initial value the method of moments estimator for $\alpha$, which is

$$\hat{\alpha} = \frac{\bar{X}^2}{\hat{\sigma}^2}$$

```
> newton1 <- function(alpha, x) {
+     n <- length(x)
+     f <- n * log(alpha) - n * log(mean(x)) + sum(log(x)) - n *
+         digamma(alpha)
+     df <- n/alpha - n * trigamma(alpha)
+     diff <- -f/df
+     y <- alpha + diff
+     return(c(diff, y))
+ }
> Find.root <- function(alpha, x) {
+     alpha0 <- alpha
+     cnt <- 0
+     esp <- 10^(-5)
+     diff <- 1
+     while (abs(diff) > esp && cnt < 20) {
+         cnt <- cnt + 1
+         outs <- newton1(alpha, x)
+         alpha <- outs[2]
+         diff <- outs[1]
+     }
+     cat("\n Initial guess= ", alpha0, "Resulting root= ", alpha,
+         "\n")
+     return(alpha)
+ }
> alpha <- Find.root(mean(x)^2/(var(x)), x)

 Initial guess=  1.91 Resulting root=  1.98

> lambda <- alpha/mean(x)
> lambda

[1] 1.93
```

Our estimates of $\hat{\alpha}$ and $\hat{\lambda}$ are given by 1.98 and 1.93.

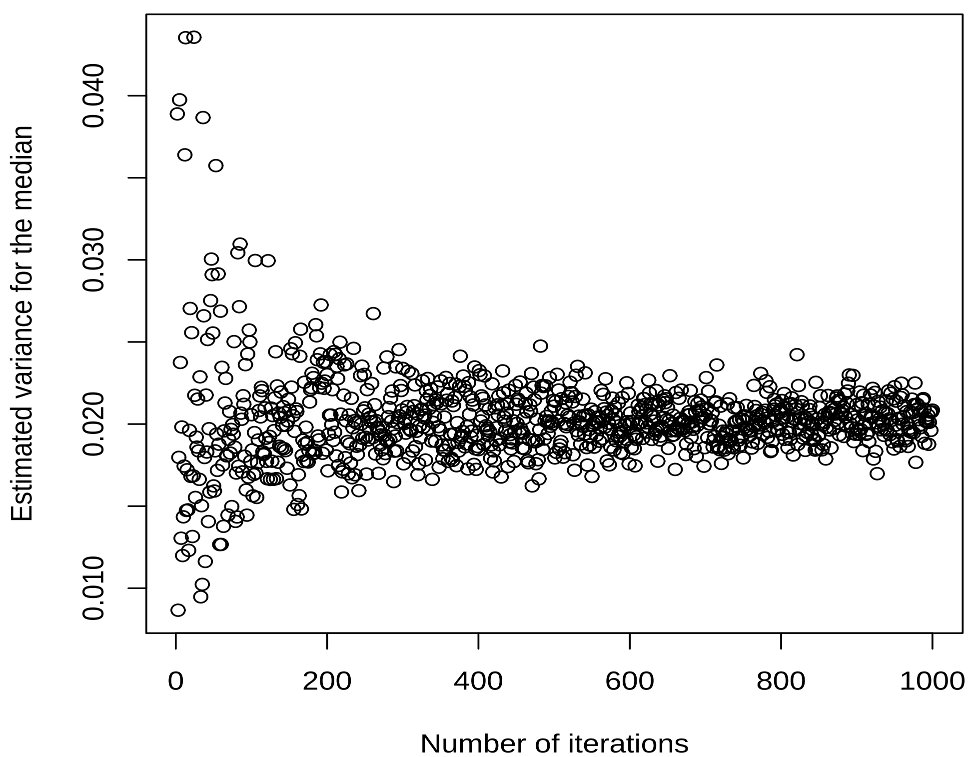
```
> alpha.vector<-rep(0,100)
> lambda.vector<-rep(0,100)
> for (i in 1:100)
+ {
+ x <- rgamma(1000, shape = alpha, rate =lambda)
+ alpha.vector[i]<-Find.root(mean(x)^2/(var(x)), x)
+ lambda.vector[i] <- alpha.vector[i]/mean(x)
+ }

> alpha.CI=c(quantile(alpha.vector,0.025),quantile(alpha.vector,1-0.025))
> lambda.CI=c(quantile(lambda.vector,0.025),quantile(lambda.vector,1-0.025))
```

We get (1.78,2.14) for $\alpha$ and (1.73,2.10) for $\lambda$.

**Question 2.**

*Answer:*

```
> data <- read.csv("lab-1.csv")
>height<-data[, "height"]
> med.std<-rep(0,1000)
> for (i in 1:1000)
+ {
+ m<-rep(0,i)
+ for (j in 1:i)
+ {
+ m[j]<-median(sample(height,30,replace=T))
+ }
+ med.std[i]=sqrt(var(m))
+ }
> plot(1:1000,med.std,xlab='Number of iterations',ylab='Estimated variance for the median')
```



Based on the plot, the standard deviation is around 0.20.