

Chapter 1

Combinatorial Analysis

A starting example: coin tossing

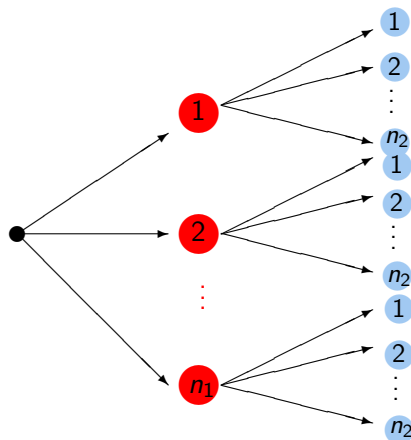
- Consider the following **random experiment**: tossing a **fair** coin twice
- There are four **possible outcomes**, namely $\{HH, HT, TH, TT\}$
- What is the **probability of the event** that we observe
 - ✦ the tail (T) on both tosses?
Answer: ...
 - ✦ at least one head (H)?
Answer: ...

To compute probability of events, need an effective method for counting the number of possible outcomes of a random experiment

$\Rightarrow$ Combinatorial Analysis

Basic Principle of Counting

Consider **two random experiments**. If experiment 1 can result in only one of n_1 possible outcomes and if, for each outcome of experiment 1, there are n_2 possible outcomes of experiment 2, then there are $n_1 \times n_2$ possible outcomes for the two experiments.



Probability
Tree Diagram

Generalized Principle of Counting

- There are a total of r experiments
- Experiment i has n_i outcomes ($i = 1, \dots, r$)
- Total number of outcomes $= n_1 \times n_2 \times \dots \times n_r$

Exercise 1

An experimenter wishes to investigate the effects of three variables - pressure, temperature and type of catalyst - on yield in a refining process. If the experimenter intends to use three settings for temperature, three settings for pressure, and two types of catalysts, how many experimental runs must be conducted to run all possible combinations of pressure, temperature and type of catalyst?

Generalized Principle of Counting (cont'd)

Exercise 2

How many different seven-digit telephone numbers can be formed if the first digit cannot be zero?

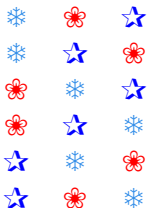
Exercise 3

How many 6-place license plates can be formed if the first 3 places are to be occupied by letters, the final 3 places by numbers and if

- (i) no repetition of letters is allowed;
- (ii) no repetition of numbers is allowed;
- (iii) no repetition of both numbers and letters is allowed.

Permutations

- Consider a set of 3 objects: {❄️, 🌸, ★}
- There are 6 different **ordered arrangements** of this set:



Each arrangement is referred to as a **permutation**

- Note: the **order** of the objects or outcomes **is important**

Permutations (cont'd)

The total number of permutations of n objects selected from n distinct objects is given by $n!$

$$n! = n \times (n - 1) \times \cdots \times 2 \times 1 \quad (1! = 1, 0! = 1)$$

The total number of permutations of r objects selected from n distinct objects is given by ${}_nP_r$ ($r \leq n$)

$${}_nP_r = n \times (n - 1) \times \cdots \times (n - r + 1) = \frac{n!}{(n - r)!}$$

Permutations - Exercises

Exercise 4

In how many ways can three executive members of a committee be assigned to three different positions: chair, treasurer, secretary?

Exercise 5

In how many ways can a committee of three be selected from among ten people if each committee member takes on a different position (chair, treasurer, secretary)?

Combinations

- Suppose we want to select different groups of r objects from a set of n distinct objects
- The order of objects in a group is **not** important here
- How many distinct subsets of size r can be formed from n distinct objects?

Example:

- There are five gift cards, each having a different design. You randomly pick two from these five cards.
- How many possible distinct subsets of two cards are there?

Solution:

- Denote the five designs as A, B, C, D, E
- The possible outcomes are

$$\{AB, AC, AD, AE, BC, BD, BE, CD, CE, DE, \\ BA, CA, DA, EA, CB, DB, EB, DC, EC, ED\}$$

and the total number of **ordered** arrangements is ...

- Here, AB and BA are two different ordered arrangements, but they form the same subset. We are interested in finding out the total number of possible distinct subsets. The order (first draw versus second draw) does not matter.
- Total number of **distinct subsets** of two cards

$$\begin{aligned} &= \frac{\text{total \# ordered arrangements of two cards from five cards}}{\text{\# of ordered arrangements of two cards from two cards}} \\ &= \end{aligned}$$

Combinations - General result

The number of subsets or **combinations** of size r that can be formed from n distinct objects ($r \leq n$) is given by

$${}_nC_r = \binom{n}{r} := \frac{n!}{r!(n-r)!} \quad (\text{Binomial coefficient})$$

Reasoning:

- Note: # ordered arrangements of r objects from n objects =
(# subsets of r objects (unordered)) $\times$ (# ways of ordering the r objects)
- That is,

$${}_nP_r = {}_nC_r \times r!$$

which gives

$${}_nC_r = \binom{n}{r} = \frac{{}_nP_r}{r!} = \frac{n \times (n-1) \times \cdots \times (n-r+1)}{r!} = \frac{n!}{(n-r)!r!}$$

The number of subsets or **combinations** of size r that can be formed from n distinct objects ($r \leq n$) is given by

$${}_nC_r = \binom{n}{r} := \frac{n!}{r!(n-r)!} \quad (\text{Binomial coefficient})$$

Gift card example (revisited):

The number of distinct subsets of two cards out of five = ...

Exercise 6

In how many ways can a committee of three be selected from among ten people?

Exercise 7

An assembly operation for a computer circuit board consists of four distinct operations, which can be performed in any order.

- (a) In how many ways can the assembly operation be performed?
- (b) One of the operations involves soldering wire to a microchip. In how many ways can this particular operation come first or second?

Multinomial coefficients

The number of ways of partitioning n distinct objects into k distinct groups containing $n_1, n_2, \dots, n_k$ objects, respectively, is

$$\frac{n!}{n_1! n_2! \cdots n_k!}, \quad \text{where } \sum_{i=1}^k n_i = n$$

Proof: From the generalized principle of counting, the number of ways of partitioning is the product:

$$\underbrace{\binom{n}{n_1}}_{\substack{\# \text{ ways} \\ \text{to form first} \\ \text{partition}}} \times \underbrace{\binom{n - n_1}{n_2}}_{\substack{\# \text{ ways} \\ \text{to form second} \\ \text{partition, given} \\ \text{the first partition}}} \times \cdots \times \underbrace{\binom{n - n_1 - \cdots - n_{k-1}}{n_k}}_{\substack{\# \text{ ways to form } k\text{-th} \\ \text{partition, given} \\ \text{the first } k - 1 \text{ partitions}}}$$

Multinomial coefficients

The number of ways of partitioning n distinct objects into k distinct groups containing $n_1, n_2, \dots, n_k$ objects, respectively, is

$$\frac{n!}{n_1!n_2!\cdots n_k!}, \quad \text{where } \sum_{i=1}^k n_i = n$$

Proof: From the generalized principle of counting, the number of ways of partitioning is the product:

$$\begin{aligned} & \underbrace{\binom{n}{n_1}}_{\substack{\# \text{ ways} \\ \text{to form first} \\ \text{partition}}} \times \underbrace{\binom{n-n_1}{n_2}}_{\substack{\# \text{ ways} \\ \text{to form second} \\ \text{partition, given} \\ \text{the first partition}}} \times \cdots \times \underbrace{\binom{n-n_1-\cdots-n_{k-1}}{n_k}}_{\substack{\# \text{ ways to form } k\text{-th} \\ \text{partition, given} \\ \text{the first } k-1 \text{ partitions}}} \\ &= \frac{n!}{n_1!(n-n_1)!} \times \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!} \times \cdots \times \frac{(n-n_1-\cdots-n_{k-1})!}{n_k! 0!} \\ &= \frac{n!}{n_1!n_2!\cdots n_k!} \end{aligned}$$

Exercise 8

A fleet of eight taxis is to be divided among three airports, A, B and C, with two going to A, five to B, and one to C.

- (a) In how many ways can this be done?
- (b) Jones drives one of the taxis. In how many ways will Jones end up at airport C?

Exercise 9

In how many ways can 8 people be seated in a row if

- (a) persons A and B must sit next to each other
- (b) there are 4 men and 4 women and no 2 men or 2 women can sit next to each other
- (c) there are 4 married couples and each couple must sit together

Chapter 1 - Learning outcomes

Aim: Demonstrate the ability to solve combinatorial problems

- Use the basic principle of counting to obtain the total number of possible outcomes in a random experiment
- Differentiate between permutations and combinations
- Explain if the order of outcomes matters in the context of the counting problem
- Apply rules on permutations and combinations in solving counting problems