R O B U S T N E S S - FALL 2015

ASSIGNMENT 2

Problem 1: Show that LMS is an S-estimate with "jump" score function:

$$\chi(y) = \begin{cases} 0 & \text{if} \quad y^2 \le c \\ 1 & \quad\quad y^2 > c. \end{cases}$$

What is the value of c?

Problem 2: Suppose that the function χ satisfies the following assumptions:
(**1**) χ is continuously differentiable
(**2**) $\chi(0) = 0$ and $\chi(\infty) = 1$.
(**3**) $\chi(-y) = \chi(y)$ for all y.
(**4**) $\chi(y)$ is non-decreasing on $[0, \infty)$.
(**5**) $0 < b = E_\Phi \chi(y) < 1$.

Show that the corresponding S-estimator, $\widehat{\boldsymbol{\beta}}$, satisfies the estimating equation:

$$\sum \psi \left(\frac{y_i - \widehat{\boldsymbol{\beta}}' \mathbf{x}_i}{\hat{\sigma}_n} \right) \mathbf{x}_i = 0$$

for an appropriate score-function function, ψ, and residuals scale $\hat{\sigma}_n$ (you must specify ψ and $\hat{\sigma}_n$).

Problem 3: Show that S-estimates are affine and regression equivariant. More precisely, set

$$\hat{\beta} = \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix}$$

and show that

(1) If $\mathbf{x} \rightarrow A\mathbf{x}$ (with A invertible) then $\hat{\beta}_1 \rightarrow A^{-1}\hat{\beta}_1$ and $\hat{\beta}_0 \rightarrow \hat{\beta}_0$

(2) If $\mathbf{x} \rightarrow \mathbf{x} + \mathbf{d}$ then $\hat{\beta}_1 \rightarrow \hat{\beta}_1$ and $\hat{\beta}_0 \rightarrow \hat{\beta}_0 - \hat{\beta}_1'\mathbf{d}$

(3) If $y \rightarrow ay$ then $\hat{\beta}_1 \rightarrow a\hat{\beta}_1$ and $\hat{\beta}_0 \rightarrow a\hat{\beta}_0$

(4) If $y \rightarrow y + a + \mathbf{x}'\mathbf{b}$ then $\hat{\beta}_1 \rightarrow \hat{\beta}_1 + \mathbf{b}$ and $\hat{\beta}_0 \rightarrow \hat{\beta}_0 + a$

Problem 4: Suppose we wish to minimize the S-scale, $S(\mathbf{b})$, which is implicitly defined by the equation

$$\frac{1}{n}\sum_{i=1}^{n} \chi\left(\frac{y_i - \mathbf{b}'\mathbf{x}_i}{s}\right) = b. \tag{1}$$

Suppose we wish to evaluate m independent candidates, $\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_m$ (a) Suppose that the current "record" after evaluating $\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_k$ is $\mathbf{b}^*$, with record value $S(\mathbf{b}^*) = s^*$. Show that $\mathbf{b}_{k+1}$ can be immediately discarded (and therefore $S(\mathbf{b}_{k+1})$ doesn't need to be computed) if

$$\frac{1}{n}\sum_{i=1}^{n} \chi\left(\frac{y_i - \mathbf{b}_{k+1}'\mathbf{x}_i}{s^*}\right) > b$$

(b) Let N be the (random) number of times equation (1) needs to be solved. Show that

$$E(N) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m} \approx \int_1^m t^{-1}dt = \log(m)$$

and

$$Var(N) = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right) - \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{m^2}\right) \tag{2}$$

Hint: The random number of evaluations, N, can be represented as

$$N = W_1 + W_2 + \cdots + W_m$$

with

$$W_i = 1 \text{ or } 0.$$

For proving equation (2) you can assume that the W_i are uncorrelated.

Problem 5: Show that Tau-estimates defined as the minimizer of

$$\tau^2\left(\boldsymbol{\beta}\right) = s_n^2\left(\boldsymbol{\beta}\right)\frac{1}{n}\sum_{i=1}^{n}\chi_2\left(\frac{r_i\left(\boldsymbol{\beta}\right)}{s_n\left(\boldsymbol{\beta}\right)}\right) \tag{3}$$

subject to

$$\frac{1}{n}\sum_{i=1}^{n}\chi_1\left(\frac{r_i\left(\boldsymbol{\beta}\right)}{s_n\left(\boldsymbol{\beta}\right)}\right) = b \tag{4}$$

satisfy an estimating equation

$$\frac{1}{n}\sum_{i=1}^{n}\Psi_n\left(\frac{r_i\left(\mathbf{t}\right)}{s_n\left(\mathbf{t}\right)}\right)\mathbf{x}_i = 0$$

with "adaptive" score function Ψ_n, which depends on the data and has the form

$$\Psi_n\left(y\right) = w_n\psi_1\left(y\right) + \left(1 - w_n\right)\psi_2\left(y\right),$$

where $0 < w_n < 1$ is a weight that depends on the data. When is w_n close to 1? When is w_n close to 0?

Problem 6: Consider the function

$$g(y, s) = s^2\chi(y/s). \tag{5}$$

What can you say about the monotonicity (in s) of $g(y, s)$ for given y. Give examples where this monotonicity is (is not) satisfied. In particular, study (5) for the cases of Tukey's and Huber's families of loss functions.

Problem 7: Suppose that the function χ_2 in (3)-(4) is such that $g(y, s)$ given by (5) with $\chi = \chi_2$ is non-decreasing in s for all y. Consider the following "record step" for an algorithm to compute the Tau-estimate via re-sampling:

Record Step: Let β^*, τ^* and s^* be the current values corresponding to the last observed "record". Let β is the next "candidate" to be evaluated for a possible new record. Show that β can be discarded if the following two conditions hold:

(1) $\quad \frac{1}{n}\sum_{i=1}^{n}\chi_1\left(\frac{r_i(\beta)}{s^*}\right) > b.$

(2) $\quad \frac{1}{n}\sum_{i=1}^{n}\chi_2\left(\frac{r_i(\beta)}{s^*}\right) > \frac{1}{n}\sum_{i=1}^{n}\chi_2\left(\frac{r_i(\beta^*)}{s^*}\right)$

Let N be the number of times condition (1) or condition (2) above are violated. Compute (numerically) mean and the standard deviation of N. Notice that $E(N)$ and $SD(N)$ determine the overall expected computational burden for the proposed resampling algorithm.