STAT 321 / ELEC 321

HOMEWORK 2-2

Problem 1: Let (X, Y) be jointly normal with

$$\mu = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} 4 & 2 \\ 2 & 9 \end{pmatrix}$$

(a) Calculate the correlation coefficient, ρ.

(b) What is your prediction for Y if you know that $X = 5$? What is the probability that your absolute prediction error is less 0.2.

(c) What percentage of the variance of the variance of Y is explained by X?

Problem 2: Suppose that $X \sim N(0, 1)$, $W \sim Bin(1, 1/2)$ and W, X are independent. Set

$$Y = 2(W - 1/2)X$$

Show that

(a) Show that $Cov(X, Y) = 0$. **Hint:** Use that $Cov(X, Y) = E(XY) - E(X)E(Y)$ and that $E(XY) = E[E(XY|W)]$.

(b) Show that $Y \sim N(0, 1)$. **Hint:** Notice that $P(Y \leq y) = E(I(Y \leq y))$ where $I(Y \leq y) = 1$ if $Y \leq y$ and
$I(Y \leq y) = 0$ otherwise.

(c) Show that X and Y are not independent. **Hint:** Calculate $P(Y > 1)$ and $P(Y > 1| - 1/2 < X < 1/2)$ and draw the desired conclusion from the results of your calculations.

(d) We have learned that if (V, U) have joint normal distribution and $Cov(U, V) = 0$ then U and V are independent. Why (c) doesn't contradict (a)?

(e) Can two random variable have marginal normal distributions but not be jointly normal?

Problem 3: Suppose that the random vector

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix}$$

has multivariate normal distribution with

$$\mu = \begin{pmatrix} 2 \\ 5 \\ 3 \\ 1 \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} 5 & 2 & 0 & 1 \\ 2 & 9 & 6 & 5 \\ 0 & 6 & 5 & 3 \\ 1 & 5 & 3 & 4 \end{pmatrix}$$

and set

$$\mathbf{X}_1 = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \quad \text{and} \quad \mathbf{X}_2 = \begin{pmatrix} X_3 \\ X_4 \end{pmatrix}$$

(a) Show that

$$Y = \frac{X_1 + X_2 + X_3 + X_4}{4}$$

is also normal with mean μ_Y and variance σ_Y^2. What are μ_Y and σ_Y^2 ?

(b) What is the marginal distribution of $\mathbf{X}_2$?

(c) What is the conditional distribution of $\mathbf{X}_2$ given

$$\mathbf{X}_1 = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 2.3 \\ 4.8 \end{pmatrix}?$$

(c) Find the joint distribution for

$$\mathbf{Y} = \begin{pmatrix} X_1 - X_2 \\ X_2 - X_3 \\ X_3 - X_4 \end{pmatrix}$$

Hint: Notice that

$$\mathbf{Y} = \begin{pmatrix} X_1 - X_2 \\ X_2 - X_3 \\ X_3 - X_4 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix}$$

Problem 4: The random variables X and Y have a bivariate normal distribution, with mean $\mu_X = \mu_Y = 0$, $\sigma_X^2 = \sigma_Y^2 = 2$ and $Cov\,(X, Y) = -1$.

(a) What is the correlation coefficient ρ between X and Y?

(b) Find the best linear predictor $\widehat{Y} = a_0 + b_0 X$. That is, find a_0 and b_0 such that

$$E\,(Y - a - bX)^2 \geq E\,(Y - a_0 - b_0 X)^2\,, \quad \text{for all } a, b$$

(c) Compare $\widehat{Y}$ with the $E\,(Y|X)$. What do you conclude?

(d) What is your prediction for Y given $X = 1.3$?

(e) What fraction of the variance of Y is explained by X?