# Review for the Midterm

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OUTLINE

# PROBABILITY

- Probabilities calculations using simple formulas
  - $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (union)
  - $P(A|B) = P(A \cap B) / P(B)$ (conditional)
  - $P(A \cap B) = P(A) P(B|A) = P(B) P(A|B)$ (multiplication)
  - $P(A) = P(A \cap B) + P(A \cap B^c)$

    $= P(B) P(A|B) + P(B^c) P(A|B^c)$ (total probability)
  - Bayes' formula
  - Counting, combinatorial calculations

- Independence
  - $P(A \cap B) = P(A)\,P(B)$
  - Systems of independent components
  - Reliability calculations

# RANDOM VARIABLES

- Discrete random variables
  - pmf, cdf, mean, variance, moment generating function
  - Binomial, Poisson, geometric
- Continuous random variables
  - Density function, cdf, mean, variance, moment generating function
  - Uniform, exponential
  - Relation between exponential and Poisson random variables
  - Functions of continuous random variables
  - Failure rate
  - Median

# NORMAL RANDOM VARIABLES

- Standard normal
  - Symmetry, absolute value
  - Probabilities, quantiles
- General normal:
  - Symmetry, Standardization
  - Probabilities, quantiles
- "Word problems" using normal random variables

S O M E
P R A C T I C E
R O B L E M S

# Probability Calculation

**Problem:** Suppose that $P(A) = 0.70$, $P(B|A) = 0.40$, and $P(B|A^c) = 0.30$. Calculate

**(a)** $P(A|B)$ **(b)** $P(A \cup B)$

**Solution (a)**

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B|A)\,P(A) + P(B|A^c)\,P(A^c)}$$

$$= \frac{0.40 \times 0.70}{0.40 \times 0.70 + 0.30 \times 0.30} = 0.756\,76$$

**Solution (b)**

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\begin{aligned} P(A) &= 0.70 \quad \text{(given)} \\ P(A \cap B) &= P(A)\,P(B|A) = 0.70 \times 0.40 = 0.28 \\ P(B) &= P(A \cap B) + P(A^c \cap B) = 0.28 + P(B|A^c)\,P(A^c) \\ &= 0.28 + 0.30(1 - 0.70) = 0.37 \end{aligned}$$

Hence

$$P(A \cup B) = 0.70 + 0.37 - 0.28 = 0.79$$

# Normal Distribution

**Problem:** In an industrial process the diameter (in centimeters) of a ball bearing is a normal random variable with mean $\mu$ equal to the intended diameter and standard deviation $\sigma$ *cm*. For what values of $\sigma$ is the difference between the intended and actual diameter less than 0.5 *cm* with probability 0.90 or more? You can use that $2\Phi(1.96) - 1 = 0.95$ and/or that $2\Phi(1.645) - 1 = 0.90$

**Solution**

$$X = \text{ball bearing diameter} \sim N\left(\mu, \sigma^2\right)$$

We want $\quad 0.90 \leq P\left(|X-\mu| < 0.5\right).$

$$\begin{aligned} 2\Phi\left(1.645\right) - 1 &= 0.90 \leq P\left(\left|\frac{X-\mu}{\sigma}\right| < \frac{0.5}{\sigma}\right) = P\left(|Z| < \frac{0.5}{\sigma}\right) \\ &= 2\Phi\left(\frac{0.5}{\sigma}\right) - 1 \end{aligned}$$

# Normal Distribution (continued)

$$\Rightarrow \quad \Phi\left(\frac{0.5}{\sigma}\right) \geq \Phi\left(1.645\right) \Rightarrow \frac{0.5}{\sigma} \geq 1.645 \Rightarrow \frac{0.5}{1.645} \geq \sigma$$

$$\Rightarrow \quad \sigma \leq \frac{0.5}{1.645} = 0.303\,95$$

**Problem:** Suppose that the time (in months) to failure of a communication system has failure rate

$$\lambda(x) = 3x^2, \qquad x > 0$$

**(a)** Calculate the probability that the system will survive 1 month.

**(b)** What is the median life for the system?

**(c)** What is the expected life for the system? **NOTE:** an answer in terms of an integral would be sufficient

**Solution:**

$$\Lambda(x) = \int_0^x \lambda(t)\,dt = 3\int_0^x t^2 dt = t^3\Big|_0^x = x^3$$

$$F(x) = 1 - e^{-\Lambda(x)} = 1 - e^{-x^3}$$

**(a)**

$$P(X > 1) = 1 - \left[1 - e^{-1^3}\right] = e^{-1} = 0.367\,88$$

**(b)**

$$1 - e^{-x^3} = 0.5 \Rightarrow e^{-x^3} = 0.5 \Rightarrow -x^3 = \ln(0.5) = -\ln(2)$$

$$\Rightarrow m = \sqrt[3]{\ln(2)} = 0.885$$

**(c)**

$$f(x) = F'(x) = 3x^2 e^{-x^3}$$

$$E(X) = 3\int_0^\infty x^3 e^{-x^3} dx$$

The integral is not easy to calculate analytically. using R I found:

$$E(X) = 3\int_0^\infty x^3 e^{-x^3} dx = 0.89298$$

```
> func = function(x) {  y = 3 * x^3 * exp(-x^3) ; return(y)  }
> integrate( func, 0, Inf )
```