Module 1: Probability

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RANDOM EXPERIMENT

- The outcome cannot be determined beforehand
- Examples
  - number of shots needed to decide a tennis point
  - yield of a chemical process
  - max-wind speed in Vancouver in 2015
  - your final grade in ELEC 321

SAMPLE SPACE

- List of possible outcomes of a random experiment
- Denoted by Ω
- Examples
  - number of shots: $\Omega = \{1, 2, 3, 4, ...\}$
  - yield of a chemical process: $\Omega = [0, 100]$ in percentage
  - max-wind speed: $\Omega = [0, \infty)$ [or $[0, 1000)$ in km/hour]
  - final grade: $\Omega = [0, 100]$

EVENT

- Subsets of Ω are called events
- Denoted by A, B, C,etc
- A occurs if $\omega \in A$
- Examples
  - $A = \{\text{at most 3 shots}\} = \{1, 2, 3\}$
  - $B = \{\text{between 20 and 40 percent}\} = [20, 40]$
  - $C = \{\text{over 100 km/hour}\} = [100, \infty)$
  - $D = \{\text{an A}\} = [80, 100]$
- Set operations: union $\cup$, intersection $\cap$, complement

Sigma Fields

- A collection of events
- Formal definition of σ-field
  - ϕ and $\Omega \in \mathcal{F}$
  - $A \in \mathcal{F} \implies A^c \in \mathcal{F}$
  - $A_n \in \mathcal{F} \implies \cup_{n=1}^{\infty} A_n \in \mathcal{F}$
- The domain for a probability function (needed for technical reasons)
- Examples of σ-fields
  - $\{\phi, \Omega\}$
  - $\{A, A^c, \phi, \Omega\}$
  - all subsets of Ω

DISCUSSION

- $A_n \in \mathcal{F} \implies \cap_{n=1}^{\infty} A_n \in \mathcal{F}$ (show this)
- Intersection of σ-fields is also a σ-field (show this)
- σ-field generated by $\mathcal{C}$, $\mathcal{F}(\mathcal{C})$,
- Borel sets $= \mathcal{F}$ (open intervals)

PROBABILITY FUNCTION

$P : \mathcal{F} \rightarrow [0, 1]$ satisfying:

1. $P(\Omega) = 1$
2. $P(A) \geq 0$, for all $A \in \mathcal{F}$
3. A_n disjoint $\implies P\left(\cup_{n=1}^{\infty} A_n\right) = \Sigma_{n=1}^{\infty} P(A_n)$

Probability space: $(\Omega, \mathcal{F}, \ P)$

SIMPLE RESULT

$$P(A^c) = 1 - P(A)$$

Proof:

$$A^c \cup A = \Omega$$

$$\Longrightarrow \qquad P(A^c) + P(A) = P(\Omega) = 1$$

$$P(A^c) = 1 - P(A)$$

SIMPLE RESULT

$$A \subset B \implies P(A) \leq P(B)$$

Proof:

$$A \cup A^c = \Omega$$

$$B \cap (A \cup A^c) = B \cap \Omega = B$$

$$(B \cap A) \cup (B \cap A^c) = A \cup (B \cap A^c) = B$$

$$P[A \cup (B \cap A^c)] = P(A) + P(B \cap A^c) = P(B)$$

$$\implies \quad P(A) \leq P(B)$$

SIMPLE RESULT

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof:

$$A \cup B = A \cup (B \cap A^c)$$

$$\Longrightarrow P(A \cup B) = P(A) + P(B \cap A^c) \qquad (1)$$

On the other hand,

$$P(B) = P(B \cap A^c) + P(B \cap A)$$

$$\Longrightarrow P(B \cap A^c) = P(B) - P(B \cap A) \qquad (2)$$

From (1) and (2)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

SIMPLE RESULT

- Boole's Inequality

$$P\left(\cup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P\left(A_i\right)$$

- If the events are disjoint, then the inequality becomes an equality.

- Easy to prove for the case of $n = 2$

- For general n use induction

EQUALLY LIKELY OUTCOMES

$$P(A) = \frac{\#\ \text{of outcomes in } A}{\#\ \text{outcomes in } \Omega}$$

EXAMPLE - LOTTERY 6/49

- Players choose 6 numbers between 1 and 49
- 6 numbers are randomly drawn
- Prize depends on the number of matched numbers
- What is the probability of matching exactly x randomly drawn numbers?

EXAMPLE - LOTTERY 6/49 (continued)

- **Combinatorial Formula:** in how many ways we can choose k items from a set of n items?

- $$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!\,(n-k)!}$$

- $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1.$

  For example $4! = 4 \times 3 \times 2 \times 1 = 24$

- **Convention:** $0! = 1$

- Matching exactly x numbers and missing the other $6 - x$ numbers
- **Mind Experiment:** Consider a box with 49 balls numbered 1 to 49. Six of these balls are labeled "W" (your six chosen numbers), the remaining 43 are labeled "L" (the non chosen numbers).
- Formula:

$$p(x) = \frac{\binom{6}{x}\binom{43}{6-x}}{\binom{49}{6}}$$

RESULTS

x	$p(x)$
0	0.43596
1	0.41302
2	0.13238
3	0.01765
4	0.00097
5	0.00002
6	0.00000

In fact, $p(6) = 0.0000000715$