Module 6: Multivariate Normal

Ruben Zamar
Department of Statistics
UBC

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STANDARD MULTIVARIATE NORMAL N(**0**,**I**)

- Suppose that $Z_1, Z_2, ..., Z_p$ are independent standard normal random variables

- Their joint density is

$$
\begin{aligned}
f(z_1, z_2, ..., z_p) &= \varphi(z_1) \times \varphi(z_2) \times \cdots \times \varphi(z_p) \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z_1^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z_2^2} \times \cdots \times \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z_p^2} \\
&= (2\pi)^{-p/2} e^{-\frac{1}{2}\left(z_1^2 + z_2^2 + \cdots + z_p^2\right)} \\
&= (2\pi)^{-p/2} e^{-\frac{1}{2}\mathbf{z}'\mathbf{z}}
\end{aligned}
$$

NOTATION

$$\mathbf{z}'\mathbf{z} = (z_1, z_2, ..., z_p) \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_p \end{pmatrix}$$

$$= z_1^2 + z_2^2 + \cdots + z_p^2$$

MEAN

$$\boldsymbol{\mu} = \mathbf{E}(\mathbf{Z}) = \begin{pmatrix} E(Z_1) \\ E(Z_2) \\ \vdots \\ E(Z_p) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \mathbf{0}$$

COVARIANCE MATRIX

$$\Sigma = Cov\left(\mathbf{Z}\right) = \begin{pmatrix} E\left(Z_1^2\right) & E\left(Z_1 Z_2\right) & \cdots & E\left(Z_1 Z_p\right) \\ E\left(Z_2 Z_1\right) & E\left(Z_2^2\right) & \cdots & E\left(Z_2 Z_p\right) \\ \cdots & \cdots & \cdots & \cdots \\ E\left(Z_p Z_1\right) & E\left(Z_p Z_2\right) & \cdots & E\left(Z_p^2\right) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & & & \\ 0 & 0 & \cdots & 1 \end{pmatrix} = I$$

MULTIVARIATE NORMAL

- Suppose $\mathbf{Z} \sim \mathbf{N}(\mathbf{0}, I)$.

- Define a new random vector

$$\mathbf{X} = A\mathbf{Z} + \mathbf{b}$$

where A is a $p \times p$, invertible matrix.

- Therefore

$$\mathbf{Z} = A^{-1}(\mathbf{X} - \mathbf{b})$$

GENERAL MULTIVARIATE NORMAL

- Mean

$$\mu = E(\mathbf{X}) = E(A\mathbf{Z} + \mathbf{b})$$

$$= AE(\mathbf{Z}) + \mathbf{b} = \mathbf{b}$$

where A is a $p \times p$, invertible matrix.

- Covariance

$$\Sigma = Cov(\mathbf{X}) = Cov(A\mathbf{Z} + \mathbf{b})$$

$$= Cov(A\mathbf{Z}) = ACov(\mathbf{Z})A' = AA'$$

GENERAL MULTIVARIATE NORMAL

- The Jacobian for this transformation is

$$J = \left|\det\left(A^{-1}\right)\right|$$

$$= \frac{1}{\left|\det\left(A\right)\right|}$$

GENERAL MULTIVARIATE NORMAL

Hence

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{|\det(A)|}\ f_{\mathbf{Z}}\left(A^{-1}(\mathbf{x}-\mathbf{b})\right)$$

$$= \frac{1}{|\det(A)|}\frac{1}{(2\pi)^{p/2}}e^{-\frac{1}{2}(\mathbf{x}-\mathbf{b})'\left(A^{-1}\right)'A^{-1}(\mathbf{x}-\mathbf{b})}$$

$$= \frac{1}{\det(\Sigma)^{1/2}}\frac{1}{(2\pi)^{p/2}}e^{-\frac{1}{2}(\mathbf{x}-\mathbf{b})'\Sigma^{-1}(\mathbf{x}-\mathbf{b})}$$

GENERAL MULTIVARIATE NORMAL

Notice that

$$\Sigma = AA' \Rightarrow \Sigma^{-1} = \left(A'\right)^{-1} A^{-1} = \left(A^{-1}\right)' A^{-1}$$

Also

$$\det\left(\Sigma\right) = \det\left(AA'\right) = \det\left(A\right)\det\left(A'\right) = \det\left(A\right)^{2}$$

$$\Rightarrow \det\left(\Sigma\right)^{1/2} = \left|\det\left(A\right)\right|$$

PROPERTIES OF THE MULTIVARIATE NORMAL

1. Linear transformation of normal vectors are normal
   Suppose $\mathbf{X} \sim N(\mu, \Sigma)$ and the $q \times p$ matrix C has full rank. Then

$$\mathbf{Y} = C\mathbf{X} + \mathbf{d} \sim N\left(C\mu + \mathbf{d}, C\Sigma C'\right)$$

2. Marginal distribution are normal
   Suppose that

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}\right)$$

Then

$$\mathbf{X}_1 \sim N(\mu_1, \Sigma_{11}) \quad \text{and} \quad \mathbf{X}_2 \sim N(\mu_2, \Sigma_{22})$$

PROPERTIES OF THE MULTIVARIATE NORMAL

3. Conditional distributions are normal
Suppose that

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \right)$$

Then

$$\mathbf{X}_1 | \mathbf{X}_2 = \mathbf{x}_2 \ \sim \ N\left(\boldsymbol{\mu}_{1.2}, \Sigma_{1.2} \right)$$

with

$$\boldsymbol{\mu}_{1.2} = \boldsymbol{\mu}_1 + \Sigma_{12} \Sigma_{22}^{-1} \left(\mathbf{x}_2 - \boldsymbol{\mu}_2 \right)$$

$$\Sigma_{1.2} = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}$$

EXAMPLE

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N\left(\begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 2 \\ 1 & 4 & 3 \\ 2 & 3 & 5 \end{pmatrix} \right)$$

- $X_1 \sim N(2,2)$, $X_2 \sim N(5,4)$, $X_3 \sim N(1,5)$,

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N\left(\begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \right)$$

$$\begin{pmatrix} X_1 \\ X_3 \end{pmatrix} \sim N\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix} \right)$$

$$\begin{pmatrix} X_2 \\ X_3 \end{pmatrix} \sim N\left(\begin{pmatrix} 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 3 & 5 \end{pmatrix} \right)$$

EXAMPLE (continued)

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N\left(\begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 2 \\ 1 & 4 & 3 \\ 2 & 3 & 5 \end{pmatrix}\right)$$

$$X_1 + X_2 - X_3 \sim N\left(2+5-1\ ,\ \ (1,1,-1)\begin{pmatrix} 2 & 1 & 2 \\ 1 & 4 & 3 \\ 2 & 3 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ \text{-1} \end{pmatrix}\right)$$

$$X_1 + X_2 - X_3 \sim N\,(6, 3)$$

EXAMPLE (continued)

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N\left(\begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 2 \\ 1 & 4 & 3 \\ 2 & 3 & 5 \end{pmatrix}\right)$$

$$X_1 | X_2 = 4 \sim N\left(2 + \frac{1}{4}(4-5)\ ,\ 2 - \frac{1}{4}\right)$$

$$X_1 | X_2 = 4 \sim N(1.75\ ,\ 1.75)$$

EXAMPLE (continued)

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N\left(\begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 2 \\ 1 & 4 & 3 \\ 2 & 3 & 5 \end{pmatrix} \right)$$

$X_1 | X_2 = 6, X_3 = 0$ **?**

$$\mu_{1.2} = 2 + \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 3 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 6 - 5 \\ 0 - 1 \end{pmatrix} = 1.454545$$

$$\Sigma_{1.2} = 2 - \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 3 & 5 \end{pmatrix}^{-1} \begin{matrix} 1 \\ 2 \end{matrix} = 1.181818$$