Module 6: Multivariate Normal

Ruben Zamar Department of Statistics UBC

January 14, 2015

STANDARD MULTIVARIATE NORMAL N(0,I)

- Suppose that Z₁, Z₂, ..., Z_p are independent standard normal random variables
- Their joint density is

$$f(z_1, z_2, ..., z_p) = \varphi(z_1) \times \varphi(z_2) \times \cdots \times \varphi(z_p)$$

$$=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z_1^2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z_2^2}\times\cdots\times\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z_p^2}$$

$$= (2\pi)^{-p/2} e^{-\frac{1}{2}(z_1^2 + z_2^2 + \dots + z_p^2)}$$

$$= (2\pi)^{-p/2} e^{-\frac{1}{2}\mathbf{z}'\mathbf{z}}$$

$$\mathbf{z}'\mathbf{z} = (z_1, z_2, ..., z_p) \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_p \end{pmatrix}$$

$$= z_1^2 + z_2^2 + \cdots + z_p^2$$

3

・ロト ・ 日 ト ・ ヨ ト ・ ヨ ト

$$\boldsymbol{\mu} = \mathbf{E} \left(\mathbf{Z} \right) = \begin{pmatrix} E \left(Z_1 \right) \\ E \left(Z_2 \right) \\ \vdots \\ E \left(Z_p \right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \mathbf{0}$$

Ruben Zamar Department of Statistics

January 14, 2015 4 / 16

3

・ロト ・ 日 ト ・ ヨ ト ・ ヨ ト

$$\Sigma = Cov \left(\mathbf{Z}\right) = \begin{pmatrix} E\left(Z_1^2\right) & E\left(Z_1Z_2\right) & \cdots & E\left(Z_1Z_p\right) \\ E\left(Z_2Z_1\right) & E\left(Z_2^2\right) & \cdots & E\left(Z_2Z_p\right) \\ \cdots & \cdots & \cdots \\ E\left(Z_pZ_1\right) & E\left(Z_pZ_2\right) & \cdots & E\left(Z_p^2\right) \end{pmatrix}$$

$$= \left(\begin{array}{rrrrr} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & & & & \\ 0 & 0 & \cdots & 1 \end{array}\right) = I$$

Ruben Zamar Department of Statistics

3 🕨 🖌 3

æ

MULTIVARIATE NORMAL

- Suppose $\mathbf{Z} \sim \mathbf{N}(\mathbf{0}, I)$.
- Define a new random vector

$$\mathbf{X} = A\mathbf{Z} + \mathbf{b}$$

where A is a $p \times p$, invertible matrix.

Therefore

$$\mathbf{Z} = A^{-1} \left(\mathbf{X} - \mathbf{b} \right)$$

GENERAL MULTIVARIATE NORMAL

Mean

$$\mu = E(\mathbf{X}) = E(A\mathbf{Z} + \mathbf{b})$$

$$=AE(\mathbf{Z})+\mathbf{b}=\mathbf{b}$$

where A is a $p \times p$, invertible matrix.

• Covariance

$$\Sigma = \mathit{Cov}\left(\mathbf{X}
ight) = \mathit{Cov}\left(\mathit{A}\mathbf{Z} + \mathbf{b}
ight)$$

$$= Cov(A\mathbf{Z}) = ACov(\mathbf{Z})A' = AA'$$

• The Jacobian for this transformation is

$$egin{aligned} J &= \left| \mathsf{det} \left(A^{-1}
ight)
ight| \ &= rac{1}{\left| \mathsf{det} \left(A
ight)
ight|} \end{aligned}$$

GENERAL MULTIVARIATE NORMAL

Hence

$$f_{\mathbf{X}}\left(\mathbf{x}
ight) = rac{1}{\left|\det\left(A
ight)
ight|} f_{\mathbf{Z}}\left(A^{-1}\left(\mathbf{x}-\mathbf{b}
ight)
ight)$$

$$=\frac{1}{\left|\det\left(A\right)\right|}\frac{1}{\left(2\pi\right)^{p/2}}e^{-\frac{1}{2}\left(\mathbf{x}-\mathbf{b}\right)'\left(A^{-1}\right)'A^{-1}\left(\mathbf{x}-\mathbf{b}\right)}$$

$$=\frac{1}{\det\left(\Sigma\right)^{1/2}}\frac{1}{\left(2\pi\right)^{p/2}}e^{-\frac{1}{2}\left(\mathbf{x}-\mathbf{b}\right)'\Sigma^{-1}\left(\mathbf{x}-\mathbf{b}\right)}$$

э

Image: A mathematical states and a mathem

GENERAL MULTIVARIATE NORMAL

Notice that

$$\Sigma = \mathcal{A}\mathcal{A}' \Rightarrow \Sigma^{-1} = \left(\mathcal{A}'
ight)^{-1}\mathcal{A}^{-1} = \left(\mathcal{A}^{-1}
ight)'\mathcal{A}^{-1}$$

Also

$$det (\Sigma) = det (AA') = det (A) det (A') = det (A)^{2}$$
$$\Rightarrow det (\Sigma)^{1/2} = |det (A)|$$

Ruben Zamar Department of Statistics

∃ ► < ∃ ►</p>

Image: A matrix and a matrix

æ

PROPERTIES OF THE MULTIVARIATE NORMAL

 Linear transformation of normal vectors are normal Suppose X ~N (μ,Σ) and the q × p matrix C has full rank. Then

$$\mathbf{Y} = C\mathbf{X} + \mathbf{d} \sim N(C\mu + \mathbf{d}, C\Sigma C')$$

 Marginal distribution are normal Suppose that

$$\mathbf{X} = \left(\begin{array}{c} \mathbf{X}_1 \\ \mathbf{X}_2 \end{array}\right) \sim N\left(\left(\begin{array}{c} \mu_1 \\ \mu_2 \end{array}\right), \left(\begin{array}{c} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{array}\right)\right)$$

Then

$$\mathbf{X}_{1} \sim \mathcal{N}\left(\boldsymbol{\mu}_{1}, \boldsymbol{\Sigma}_{11}\right)$$
 and $\mathbf{X}_{2} \sim \mathcal{N}\left(\boldsymbol{\mu}_{2}, \boldsymbol{\Sigma}_{22}\right)$

PROPERTIES OF THE MULTIVARIATE NORMAL

3. Conditional distributions are normal Suppose that

$$\mathbf{X} = \left(egin{array}{c} \mathbf{X}_1 \ \mathbf{X}_2 \end{array}
ight) \sim \mathcal{N} \left(\left(egin{array}{c} \mu_1 \ \mu_2 \end{array}
ight), \left(egin{array}{c} \Sigma_{11} & \Sigma_{12} \ \Sigma_{21} & \Sigma_{22} \end{array}
ight)
ight)$$

Then

$$\mathbf{X}_1 | \mathbf{X}_2 = \mathbf{x}_2 ~ \sim ~ \mathcal{N}\left(\boldsymbol{\mu}_{1.2}, \boldsymbol{\Sigma}_{1.2}
ight)$$

with

$$\mu_{1.2} = \mu_1 + \Sigma_{12} \Sigma_{22}^{-1} \left(\mathbf{x}_2 - \mu_2 \right)$$

$$\Sigma_{1.2} = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}$$

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N \left(\begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 2 \\ 1 & 4 & 3 \\ 2 & 3 & 5 \end{pmatrix} \right)$$

• $X_1 \sim N (2, 2), X_2 \sim N (5, 4), X_3 \sim N (1, 5),$

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N \left(\begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \right)$$

$$\begin{pmatrix} X_1 \\ X_3 \end{pmatrix} \sim N \left(\begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \right)$$

$$\begin{pmatrix} X_1 \\ X_3 \end{pmatrix} \sim N \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix} \right)$$

$$\begin{pmatrix} X_2 \\ X_3 \end{pmatrix} \sim N \left(\begin{pmatrix} 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 3 & 5 \end{pmatrix} \right)$$

Ruben Zamar Department of Statistics

メロト メポト メヨト メヨト

3

EXAMPLE (continued)

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim \mathcal{N} \left(\begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 2 \\ 1 & 4 & 3 \\ 2 & 3 & 5 \end{pmatrix} \right)$$

$$X_1 + X_2 - X_3 \sim N\left(2 + 5 - 1, (1, 1, -1)\left(\begin{array}{rrr}2 & 1 & 2\\1 & 4 & 3\\2 & 3 & 5\end{array}\right)\left(\begin{array}{r}1\\1\\-1\end{array}\right)\right)$$

 $X_1 + X_2 - X_3 \sim N(6,3)$

æ

EXAMPLE (continued)

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim \mathcal{N} \left(\begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 2 \\ 1 & 4 & 3 \\ 2 & 3 & 5 \end{pmatrix} \right)$$

$$X_1|X_2=4 \sim N\left(2+rac{1}{4}\left(4-5
ight)$$
 , $2-rac{1}{4}
ight)$

$$X_1|X_2 = 4 \sim N(1.75, 1.75)$$

Ruben Zamar Department of Statistics

January 14, 2015 15 / 16

æ

< □ > < ---->

EXAMPLE (continued)

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim \mathcal{N} \left(\begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 2 \\ 1 & 4 & 3 \\ 2 & 3 & 5 \end{pmatrix} \right)$$

$$X_1 | X_2 = 6, X_3 = 0$$
 ?

$$\mu_{1.2} = 2 + \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 3 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 6-5 \\ 0-1 \end{pmatrix} = 1.454545$$

$$\Sigma_{1.2} = 2 - \left(egin{array}{cccc} 1 & 2 \end{array}
ight) \left(egin{array}{cccc} 4 & 3 \ 3 & 5 \end{array}
ight)^{-1} egin{array}{ccccc} 1 & 1 \ 2 \end{array} = 1.181818$$

æ

< □ > < ---->