Problem 1: (a) An urn contains $n = 5$ chips numbered 1,2,3,4 and 5. A chip is randomly drawn. Let Y be the random variable representing the drawn number.

[5](i) Calculate the probability mass function $f(y)$ and distribution function $F(y)$ for Y.

[5] (ii) Calculate the mean and variance for Y.

(b) Two chips are sequentially drawn from the urn. The first drawn chip is put back into the urn before the second chip is drawn. Let X represent the largest of the 2 chips.

[8] (i) Complete the following table

x	$F_X(x)$	$f_X(x)$
1		
2		
3		
4		
5		

[7] (ii) Calculate the mean and the variance of X.

a) i)

y	1	2	3	4	5
$f(y)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
$F(y)$	$\frac{1}{5}$	$2/5$	$3/5$	$4/5$	1

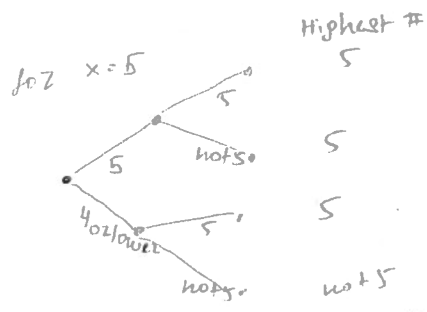
or $F_y(y) = \frac{1}{5}y$, $y = 1, \dots, 5$

ii) $\mathbb{E}Y = \sum_{i=1}^{5} y_i f(y_i) = \frac{1}{5}\sum_{i=1}^{5} i = \frac{15}{5} = 3$

$\mathbb{E}(y^2) = \sum_{i=1}^{5} y_i^2 f(y_i) = \frac{1}{5}\sum_{i=1}^{5} i^2 = \frac{55}{5} = 11$

$\mathrm{Var}\, Y = \mathbb{E}(y^2)\ \{\mathbb{E}(y)\}^2 = 11 - 3^2 = 2$

b) i)



$\mathbb{P}(x=5) = 1 - \mathbb{P}(x \neq 5) =$

$= 1 - \frac{4}{5}\cdot\frac{4}{5} = 0.36 = \frac{9}{25}$

$\mathbb{P}(x = 4 \text{ or } 5) = 1 - \mathbb{P}(x_1 \neq 4,5 \cap x_2 \neq 4,5)$

$= 1 - \frac{3}{5}\cdot\frac{3}{5} = 0.64 = 1 - \frac{9}{25} = \frac{16}{25}$

2

$\mathbb{P}(x = 3, 4 \text{ or } 5) = 1 - \mathbb{P}(x_1 = 1,2 \cap x_2 = 1,2)$

$= 1 - \frac{2}{5}\cdot\frac{2}{5} =$

$\mathbb{P}(x \geq 3) = \frac{21}{25}$

$1 - \frac{4}{25} = \frac{21}{25}$

$\mathbb{P}(x < 3) = \frac{4}{25}$

$\mathbb{P}(x = 2, 3, 4 \text{ or } 5) = 1 - \mathbb{P}(x_1 = 1 \cap x_2 = 1) = 1 - \frac{1}{}$

x	$F_X(x) = P(X \le x)$	$f_X(x)$
1	1/25	1/25
2	4/25	3/25
3	9/25	5/25
4	16/25	7/25
5	1	9/25

$//1$

$P(X=1) = P(\max\{X_1, X_2\} = 1) = P((X_1=1) \cap (X_2=1)) = \frac{1}{5^2} = \frac{1}{25}$

$P(X=5) = P(\max\{X_1, X_2\} = 5) = P((X_1=5) \cup (X_2=5)) = \frac{2}{5} - \frac{1}{25} = \frac{10-1}{25} = \frac{9}{25}$

$P(X=4) = P(\max\{X_1, X_2\} = 4) = P((X_1=4) \cap (X_2 \le 4)) + P((X_1 \le 4) \; P(X_1 = 4))$

$P(X \geq 3) = \frac{21}{25} \Rightarrow P(X \le 2) = 1 - \frac{21}{25} = \frac{4}{25}$

$P(X \geq 4) = \frac{16}{25} \Rightarrow P(X \le 3) = 1 - \frac{16}{25} = \frac{9}{25}$

$P(X \geq 5) = \frac{9}{25} \Rightarrow P(X \le 4) = 1 - \frac{9}{25} = \frac{16}{25}$

ii) $\mathbb{E}(X) = \sum_i x_i f_X(x_i) = \frac{1}{25} + 2 \cdot \frac{3}{25} + 3 \cdot \frac{5}{25} + 4 \cdot \frac{7}{25} + 5 \, \frac{9}{25} =$

$= \frac{95}{25} = 3.8$

$\mathbb{E}(X^2) = \sum_i x_i^2 f_X(x_i) = \frac{1}{25} + 4 \cdot \frac{3}{25} + 9 \, \frac{5}{25} + 16 \, \frac{7}{25} + 25 \cdot \frac{9}{25} =$

$= \frac{395}{25} = 15.8$

$Var(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = 1.36$

Problem 2: Suppose that the life time, X, of a part has failure rate (in years)

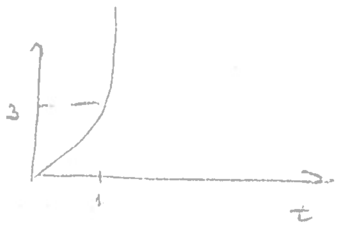
$$\lambda(t) = 3t^2, \quad \text{for} \quad t > 0.$$

[5] (a) Is the part wearing out, not changing, or getting stronger with time? Why?

[10] (b) Calculate the probability that the part will survive 6 months. **Hint:** recall that $F(x) = 1 - \exp\left\{-\int_0^x \lambda(t)\,dt\right\}$.

[10] (c) What is the half life (in years) for the part? **Hint:** the half life, m, satisfies $F(m) = 1/2$.

a)



Part is wearing out because the failure rate is an increasing f-n of time i.e. the more time has passed, the higher the p that part fails

b)

Let X : Lifetime of the part

$P(X \le x) = F_X(x) = 1 - \exp\left(-\int_0^x \lambda(t)\,dt\right)$

$P(X > \frac{1}{2}) = 1 - P(X \le \frac{1}{2}) = 1 - \left(1 - \exp\left(-\int_0^{1/2} 3t^2\,dt\right)\right) =$

$\frac{1}{2}$ because measure in years

$= \exp\left(-\int_0^{1/2} 3t^2\,dt\right) =$

$= \exp\left(-t^3 \Big|_0^{1/2}\right) = e^{-(1/2)^3} = e^{-1/}$

$= 0.882$

c) $F(m) = \frac{1}{2} \Rightarrow 1 - \exp\left\{-\int_0^x \lambda(t)\,dt\right\} = \frac{1}{2}$

$\exp\{-x^3\} = \frac{1}{2}$

$-x^3 = -\log 2$

$x^3 = \log 2$

$x = (\log 2)^{1/3} = 0.885$ years

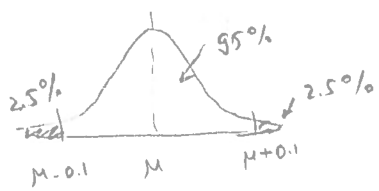
[25] **Problem 3:** The temperature reading from a thermocouple placed in a constant temperature medium is normally distributed with mean μ, the actual temperature of the medium, and standard deviation σ. For what value of σ, 95% of the readings are within 0.1^o from μ ?

Hint: Let $Z \sim N(0,1)$. $P(Z < z)$ for several values of z are given below:

z	$P(Z < z)$
1.282	0.900
1.440	0.925
1.645	0.950
1.960	0.975
2.326	0.990

X ~ temp reading $\qquad X \sim N(\mu, \sigma^2)$ $\qquad$ Need σ

s.t. $P(\mu - 0.1 \le X \le \mu + 0.1) = 95\%$



$$P(\mu - 0.1 \le X \le \mu + 0.1) = 0.95$$

$$P\left(\frac{\mu - 0.1 - \mu}{\sigma} \le X \le \frac{\mu + 0.1 - \mu}{\sigma}\right) = 0.95$$

$$P\left(-\frac{0.1}{\sigma} \le X \le \frac{0.1}{\sigma}\right) = 0.95$$

$$P\left(X \le \frac{0.1}{\sigma}\right) - P\left(X \le -\frac{0.1}{\sigma}\right) = 0.95$$

$$P\left(X \le \frac{0.1}{\sigma}\right) - \left(1 - P\left(X \le \frac{0.1}{\sigma}\right)\right) = 0.95$$

$$2 \cdot P\left(X \le \frac{0.1}{\sigma}\right) = 0.95 + 1$$

$$P\left(X \le \frac{0.1}{\sigma}\right) = 0.975 \quad \therefore \quad \frac{0.1}{\sigma} = 1.960$$

$$\Rightarrow \sigma = \frac{0.1}{1.960} = 0.051$$

Problem 4: Suppose that the joint pmf for the random vector (X, Y) is given proportional to $(x + y)$, $x = 1, 2, ..., 5$ and $y = 1, 2, ..., 5$.

[8](a) What is the joint pmf $f(x, y)$?

[9](b) Calculate $Cov(X, Y)$ **Hint:** $\sum_{x=1}^{5} x = 15$ and $\sum_{x=1}^{5} x^2 = 55$.

[8](c) Calculate $E(Y|X = 2)$

a) $f_{X,Y}(x,y) \propto x+y$ $\quad x=1,\dots,5$, $y=1\dots5$

$f_{XY}(x,y) = c\cdot(x+y)$

$\sum_x\sum_y c(x+y) = 1 \Rightarrow c\sum_{x=1}^{5}\sum_{y=1}^{5}(x+y) = 1$

$c\sum_{x=1}^{5}(5x+15) = c(5\cdot15+5\cdot15) = c\cdot150 \Rightarrow c = 1/150$

$$f_{X,Y}(x,y) = P(X=x, Y=y) = \begin{cases} \frac{x+y}{150}, & x=1,\dots,5,\ y=1,\dots,5 \\ 0, & \text{elsewhere} \end{cases}$$

b) $Cov(X,Y) = E(XY) - EX\cdot EY$

$E(XY) = \sum_{x=1}^{5}\sum_{y=1}^{5} xy\, f_{X,Y}(x,y) =$

$P(X=2) = \frac{25}{150}$

x\y	1	2	3	4	5
1	2/150	3c	4c	5c	6c
2	3c	4c	5c	6c	7c
3	4c	5c	6c	7c	8c
4	5c	6c	7c	8c	9c
5	6c	7c	8c	9c	10c

$\sum xy\, f(x,y)$	$\sum x f(x)$
70/150	20/150
170/150	50/150
300/150	90/150
460/150	140/150
650/150	200/150
1650/150	500/150

$E(X) = 500/150 = EY$

$Cov(X,Y) = \frac{1650}{150} - \left(\frac{500}{150}\right)^2 = 11 - \left(\frac{10}{3}\right)^2 = -\frac{1}{9}$

c) $f(y|X=2) = P(Y=i \mid X=2) = \frac{P(Y=i \cap X=2)}{P(X=2)} =$ $\quad P(X=2) = \frac{25}{150}$

$=$

y	$f(y\mid x=2)$
1	3/25
2	4/25
3	5/25
4	6/25
5	7/25

$E(Y|X=2) = \sum y\, f_Y(y|X=2) =$

$= \frac{85}{25} = 3.4$