Problem 1: Suppose that the life time in years of a part, X, has has distribution function

$$F(x) = 1 - e^{-x^3}, \quad \text{for} \quad x > 0.$$

[5] **(a)** Is the part wearing out, getting stronger or not changing with time? Why?

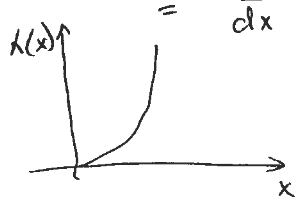
[8] **(b)** What is the median life time for the part (in years)?

[12] **(c)** Calculate the probability that the part will survive 6 extra months after serving for 1.5 years.

a) Finding failure rate given the CDF

$$\lambda(x) = -\frac{d}{dx}\ln(1 - F_X(x)) = -\frac{d}{dx}\ln\left(1-(1-e^{-x^3})\right)$$

$$= -\frac{d}{dx}(-x^3) = 3x^2$$



$\lambda(x)$ is an increasing function of x (lifetime)

$\Rightarrow$ the part is wearing out

b) $F_X(m) = \frac{1}{2} \Rightarrow \frac{1}{2} = 1 - e^{-m^3} \Rightarrow -m^3 = \ln\frac{1}{2} \Rightarrow$

$m = (\ln 2)^{\frac{1}{3}}$

c) Let X = lifetime of the part

$$P(X \geq 1.5 + 0.5 \mid X \geq 1.5) = \frac{P(X \geq 2 \cap X \geq 1.5)}{P(X \geq 1.5)}$$

$$= \frac{P(X \geq 2)}{P(X \geq 1.5)}$$

$$= \frac{1 - F_X(2)}{1 - F_X(1.5)} = \frac{e^{-2^3}}{e^{-1.5^3}} = e^{1.5^3 - 2^3}$$

[20] **Problem 2:** The controlled temperature of a room has normal distribution with mean μ and standard deviation $\sigma = 0.1^o$ Celsius. The mean μ must be set by the user. What is the smallest value of μ such that the temperature in the room is above 15^o Celsius 95% of the time?

Hint: Let $Z \sim N(0,1)$ and set $\Phi(z) = P(Z < z)$. The values of $\Phi(z)$ for several values of z are given below:

z	$\Phi(z)$
1.282	0.900
1.440	0.925
1.645	0.950
1.960	0.975
2.326	0.990

variance

Let X = temperature of a room $\qquad X \sim N(\mu, 0.1^2)$

$P(X > 15) = 0.95 \qquad \mu = ?$

$$P(X>15) = 1 - P(X \le 15)$$
$$= 1 - P\left(\frac{X-\mu}{\sigma} \le \frac{15-\mu}{\sigma}\right)$$

$\boxed{Z = \frac{X-\mu}{\sigma}}$

$$= 1 - P\left(Z \le \frac{15-\mu}{\sigma}\right) = 0.95$$
$$= 1 - \Phi\left(\frac{15-\mu}{\sigma}\right) = 0.95$$

$\boxed{\text{Recall} \quad \Phi(-a) = 1 - \Phi(a)}$

$$\Rightarrow \Phi\left(-\frac{15-\mu}{\sigma}\right) = 0.95$$

$$\Rightarrow -\frac{15-\mu}{\sigma} = 1.645$$

$$-(15-\mu) = 1.645\sigma$$
$$\mu = 15 + 1.645\sigma = 15.1645^o$$

Problem 3: Suppose that X_1 and X_2 are two independent Binomial(n, p) random variables with common mean 6 and variance 4.

[7](a) What are the values of n and p ?

[6](b) What is the distribution of $S = X_1 + X_2$? Why?

[12](c) Calculate $P(X_1 = 6 | S = 10)$

a) $X_1 \sim \text{Bin}(n, p)$ $\quad E(X_1) = 6$, $\quad Var(X_1) = 4$ $\quad n, p = ?$

Since $X_1 \sim \text{Bin}(n,p) \Rightarrow E(X_1) = np$, $Var(X_1) = np(1-p)$

$\begin{cases} np = 6 \\ np(1-p) = 4 \end{cases} \Rightarrow 1-p = \frac{4}{np} = \frac{4}{6} \Rightarrow \boxed{p = \frac{1}{3}}$

$n \cdot \frac{1}{3} = 6 \Rightarrow \boxed{n = 3 \cdot 6 = 18}$

b) $S = X_1 + X_2$

X_1 = # of successes in n indep trials with individual prob p

X_2 = # of successes in n indep trals w/ prob p

$\Rightarrow$ in case of S there are $2n$ indep trals and individual prob of success is p

$\Rightarrow S \sim \text{Bin}(2n, p)$

Another option $\quad X_1 = \sum_{i=1}^{n} y_i$ where $y_i \sim \text{Bernoulli}(p)$

$X_2 = \sum_{j=1}^{n} y_j$ where $y_j \sim \text{Bernoulli}(p)$

$\Rightarrow X_1 + X_2 = \sum_{i=1}^{2n} y_i \sim \text{Bin}(2n, p)$

c) $P(X_1 = 6 | S = 10) = \frac{P(X_1 = 6 \cap S = 10)}{P(S = 10)} = \frac{P(S=10 | X_1 = 6) \cdot P(X_1 = 6)}{P(S = 10)}$

$= \frac{P(X_1 + X_2 = 10 | X_1 = 6) \cdot P(X_1 = 6)}{P(S = 10)} = \frac{P(X_2 = 4) \cdot P(X_1 = 6)}{P(S = 10)}$

$= \frac{\binom{18}{4}\left(\frac{1}{3}\right)^4 \cdot \left(\frac{2}{3}\right)^{18-4} \cdot \binom{18}{6}\left(\frac{1}{3}\right)^6 \left(\frac{2}{3}\right)^{12}}{\binom{36}{10}\left(\frac{1}{3}\right)^{10} \cdot \left(\frac{2}{3}\right)^{26}}$

$= \frac{\binom{18}{4}\binom{18}{6}}{\binom{36}{10}}$

Problem 4: (a) A coin with probability p of "head" is independently tossed 3 times. Let Y be the position of the first head in the sequence. If no head appears in the three tosses then Y is set equal to 4.

[7] (i) Calculate the probability mass function $f_Y(y)$ and the distribution function $F_Y(y)$. as an expression of p. You are not required to simplify the expression

[6] (ii) Calculate the mean of Y ~~when $p = 1/2$~~.

[8] (iii) Five independent replications of this experiment produced the sequence 3, 3, 4, 2, 3. What is your estimate for p? **Hint:** say what equation the estimate $\widehat{p}$ must solve. No need to solve this equation.

(b) [9] The experiment described in Part (a) is independently repeated twice, yielding the sequence Y_1, Y_2. Let

$$X = \max\{Y_1, Y_2\}$$

Complete the following table

x	$F_X(x)$	$f_X(x)$
1		
2		
3		
4		

a) i) y is the position of 1st Head in a sequence of 3

$P(y=4) = P(\text{no head in 3 tosses}) = P(\text{Taie} \cap \text{Taie} \cap \text{Taee}) =$
$= (1-p)^3$

$P(y=3) = P(\text{Taie} \cap \text{Taie} \cap \text{Head}) = (1-p)^2 \cdot p$

$P(y=2) = P(\text{Taie} \cap \text{Head} \cap \text{Taie}) + P(\text{Taie} \cap \text{Head} \cap \text{Head}) =$
$= (1-p)^2 \cdot p + p^2(1-p) = p - 2p^2 + p^3 + p^2 - p^3 = p - p^2 =$
$= p(1-p)$

$P(y=1) = 1 - [P(y=2) + P(y=3) + P(y=4)]$
$= 1 - [(1-p)^3 + 2(1-p)^2 \cdot p + p^2(1-p)]$
$= 1 - (1-p)[(1-p)^2 + 2p(1-p) + p^2]$
$= 1 - (1-p)[1 - \cancel{2p} + \cancel{p^2} + \cancel{2p} - \cancel{2p^2} + \cancel{p^2}]$
$= 1 - (1-p) = p$ (makes sence, $P(y=1) = P(\text{get Heads on 1st try})$)

pmf

y_i	$P(y=y_i)$
1	p
2	$p(1-p)$
3	$p(1-p)^2$
4	$(1-p)^3$

CDF

y_i	$F_y(y_i) = P(y \le y_i)$
1	p
2	$p(1-p) + p = p(2-p)$
3	$p(2-p) + p(1-p)^2 = p(p^2 - 2p + 1 + 2 - p) = p(p^2 - 3p + 3)$
4	$p(p^2 - 3p + 3) + (1-p)^3 = 1$

5

Overflow

ii) $\mathbb{E}(y) = ?$ $\cancel{p = \frac{1}{2}}$

$$\mathbb{E}(y) = \sum_{i=1}^{4} y_i \, p(y = y_i) = \cancel{1 \cdot \frac{1}{2} + 2 \cdot \left(\frac{1}{2}\right)^2 + 3 \cdot \left(\frac{1}{2}\right)^3 + 4 \cdot \left(\frac{1}{2}\right)^3}$$

$$p + 2p(1-p) + 3p(1-p)^2 + 4(1-p)^3$$

iii) $\mathbb{E}(y)$ /without subbing in p value/

$$\mathbb{E}(y) = p + 2p(1-p) + 3p(1-p)^2 + 4(1-p)^3$$

$$\bar{y} = \frac{3+3+4+2+3}{5} = \frac{15}{5} = 3$$

$\Rightarrow$ to get $\hat{p}$ solve $\hat{p} + 2\hat{p}(1-\hat{p}) + 3\hat{p}(1-\hat{p})^2 + 4(1-\hat{p})^3 = 3$ for $\hat{p}$

/* this is method of moments estimate

b) $y = \max\{y_1, y_2\}$

Way 1 (Mathy) /But elegant/

$$F_y(y) = P(Y \le y) = P(\max(y_1, y_2) \le y)$$
$$= P(y_1 \le y \cap y_2 \le y)$$
$$= P(y_1 \le y) \cdot P(y_2 \le y) \quad \text{because } y_1 \text{ and } y_2 \text{ are independent}$$
$$= F_{y_1}(y) \cdot F_{y_2}(y)$$

Way 2 (Logical) /But equally as mathy/

$$P(y = 1) = P(y_1 = 1 \cap y_2 = 1) = P(y_1 = 1) \cdot P(y_2 = 1) = p^2$$

$$P(y = 2) = P(y_1 = 2 \cap y_2 = 1) + P(y_1 = 1 \cap y_2 = 2) + P(y_1 = 2 \cap y_2 = 2) = P(y_1 = 2) \cdot P(y_2 = 1) + P(y_1 = 1) P(y_2 \ldots + 2 \cdot P(y_1 = 2) \cdot P(y_2 = 2)$$
$$= 2p^2(1-p) + p^2(1-p)^2$$

$$P(y = 3) = P(y_1 = 3 \cap y_2 = 1) + P(y_1 = 3 \cap y_2 = 2) + P(y_2 = 3 \cap y_1 = 1) + P(y_2 = 3 \cap y_1 = \ldots + P(y_1 = 3 \cap y_2 = 3)$$
$$= 2 \cdot p^2(1-p)^2 + 2 \cdot p^2(1-p)^3 + p^2(1-p)^4$$

$$P(y = 4) = P(y_1 = 4 \cup y_2 = 4) = P(y_1 = 4) + P(y_2 = 4) - P(y_1 = 4 \cap y_2 = 4)$$
$$= 2 \cdot (1-p)^3 - (1-p)^6$$

or $P(y = 4) = 1 - [P(y = 1) + P(y = 2) + P(y = 3)]$

Overflow

pdf

y_i	$P(Y = y_i)$
1	p^2
2	$2p^2(1-p) + p^2(1-p)^2$
3	$2p^2(1-p)^2 + 2p^2(1-p)^3 + p^2(1-p)^4$
4	$2(1-p)^3 - (1-p)^6$

cdf

y_i	$F_Y(y_i) = P(Y \le y_i)$
1	p^2
2	$p^2 + 2p^2(1-p) + p^2(1-p)^2$
3	$p^2 + 2p^2(1-p) + 2p^2(1-p)^2 + 2p^2(1-p$ $+ p^2(1-p)^4$
4	1

Alternatively, as for mathy approach, can just square the CDF of y_1 or y_2