Reminder

→ Assignment 2 due Friday.

→ Midterm next Wed in Tutorial. (No Calculators)

→ OH: Mon 5-6, Tue 5-6.

→ Assignment 1 (Graded out of 130)

Anup: 11-15

Lena: 1-10

Grade Points for Assignment 1
Q 1-3, 6-10: 5pts
Q 4-5, 13,15: 10pts
Q 11,14: 15pts
Q 12: 20pts

→

Problem 1: Random variables X and Y have joint p.m.f

$$P_{X,Y}(x,y) = \begin{cases} c(x^2+y^2) & x \in \{1,2,4\}. \; y \in \{1,3\}. \\ 0 & \text{else.} \end{cases}$$

a) What is the value of c?

X \ Y	1	3
1	2c	10c
2	5c	13c
4	17c	25c

$$\sum_x \sum_y P_{X,Y}(x,y) = 1. \text{ (pmf sum to 1)}$$

$$c(2+10+5+13+17+25) = 1.$$

$$c = \frac{1}{72}.$$

b) What is the prob. $P(Y < X)$

$$Y < X \Rightarrow (2,1),\ (4,1)\ \ (4,3)$$

$X=4$
$Y=3$

$$P(Y<X) = P((2,1) \cup (4,1) \cup (4,3))$$

$$= P(2,1) + P(4,1) + P(4,3)$$

$$= 5c + 17c + 25c.$$

$$= \frac{47}{72}$$

c) $$P(Y>X) = 1 - P(Y \le X)$$

$$= 1 - [\underbrace{P(Y<X)} + P(Y=X)]$$

$$= 1 - \left[\frac{47}{72} + \frac{2}{72}\right]$$

$$= \frac{23}{72}$$

d) $$P(Y=3) \overset{TLP}{=} \sum_{x \in \{1,2,4\}} P(X=x, Y=3)$$

$$= P(1,3) + P(2,3) + P(4,3)$$

$$= 10c + 13c + 25c$$

$$= \frac{48}{72}.$$

e) Find marginal pmf's $P_X(x)$ $P_Y(y)$?

$$P_X(x) = \sum_{y} P_{X,Y}(x, y) \quad \text{[Marginalization]}$$

$x = 1.$

$$= \sum_{y \in \{1,3\}} P_{X,Y}(1, y)$$

$$= P(1,1) + P(1,3)$$

$$= 2c + 10c = \frac{12}{72}.$$

$$P_X(x) = \begin{cases} 12/72, & x = 1 \\ 18/72, & x = 2 \\ 42/72, & x = 4 \\ 0, & \text{else.} \end{cases}$$

$$P_Y(y) = \begin{cases} 48/72, & y = 3 \\ 24/72 & y = 1 \\ 0 & \text{els.} \end{cases}$$

f) $Cov(X, Y)$?

$$Cov(X, Y) = \mathbb{E}(X.Y) - \mathbb{E}(X)\,\mathbb{E}(Y)$$

$$\mathbb{E}(X) = 3 \qquad \mathbb{E}(Y) = 7/3$$

$$\mathbb{E}(XY) = \sum_{x} \sum_{y} x \cdot y \; P_{X,Y}(x, y)$$

$$= 1 \cdot 1 \; P(1,1) + 1 \cdot 3 \cdot P(1,3) + \cdots \cdots$$

$$\cdots \cdots$$

$$= 1 \cdot \frac{2}{72} + 3 \cdot \times \frac{10}{72} + 2 \times \frac{5}{72} + 6 \times \frac{13}{72} +$$

$$4 \times \frac{17}{72} + 12 \times 25/72$$

$$= \frac{488}{72}.$$

$$\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\,\mathbb{E}(Y) = -2/9.$$

9) Let A be the event $X \geq Y$. Find $\mathbb{E}(X|A)$

$$\mathbb{E}(X|A) = \sum_x x\, p_{X|A}(x)$$

$$p_{X|A}(x) \;?$$

$$p_{X|A}(x) = \frac{P(X = x, A)}{P(A)}$$

$x = 1$.

$$= \frac{P(X = 1, X \geq Y)}{P(A) \leftarrow 49/72}$$

$$= \frac{P(X=1, 1 \geq Y)}{P(A)}$$

$$= \frac{P(X=1, Y=1)}{P(A)} = \frac{2/72}{49/72}$$

$$p_{X|A}(x) = \begin{cases} 2/49, & x = 1 \\ 5/49 & x = 2 \\ 42/49, & x = 4 \\ 0 & \text{else.} \end{cases}$$

Ex: $\mathbb{E}(X|A)$ $\quad$ $\mathrm{Var}(X|A)$

2. Suppose X and Y are discrete random variables with joint p.m.f

$$f_{X,Y}(x,y) = \alpha^{x+y+2} \qquad x, y = 0, 1, \ldots \qquad 0 < \alpha < 1.$$

a) Find α ?

$$\sum_x \sum_y f_{X,Y}(x,y) = 1 \quad [\text{p.m.f sum to } 1]$$

$$\sum_x \sum_y \alpha^{x+y+2} = 1.$$

$$\cancel{\alpha^2 \sum_{x=0}^{1} \alpha^x \sum_{y=0}^{1} \alpha^y} = 1$$

$$\alpha^2 \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} \alpha^x \cdot \alpha^y = 1.$$

$$\alpha^2 \sum_{x=0}^{\infty} \alpha^x \sum_{y=0}^{\infty} \alpha^y = 1$$

$$\alpha^2 \cdot \frac{1}{1-\alpha} \cdot \frac{1}{1-\alpha} = 1.$$

$$\left(\frac{\alpha}{1-\alpha}\right)^2 = 1 \implies \alpha = \frac{1}{2}.$$

b) $\text{Cov}(X,Y) = E(XY) - E(X)\,E(Y)$

$$E(XY) = \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} x \cdot y \cdot f_{X,Y}(x,y)$$

$$= \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} x \cdot y \cdot \alpha^{x+y+2}.$$

$$= \alpha^2 \sum_{x=0}^{\infty} x \cdot \alpha^x \sum_{y=0}^{\infty} y \cdot \alpha^y .$$

$$= \alpha^2 \cdot \frac{\alpha}{(1-\alpha)^2} \cdot \frac{\alpha}{(1-\alpha)^2} .$$

$$= \frac{\alpha^4}{(1-\alpha)^4}$$

Exercise: Compute Cov(X,Y) and show that X and Y are independent

Problem 3: X and Y are independent, discrete random variables. Suppose that

$$f_X(k) = f_Y(k) = p(1-p)^k \qquad k = 0, 1, 2 \ldots \quad 0 < p < 1.$$

What is $P(X = k \mid X + Y = n)$ n

$$P(X = k \mid X + Y = n)$$

$$= \frac{P(X = k, \ X + Y = n)}{P(X + Y = n)}$$

$$= \frac{P(X = k, \ Y = n-k)}{P(X + Y = n)}$$

$$P(X = k, Y = n-k) \overset{\text{indp.}}{=} P(X = k) \ P(Y = n-k)$$

$$= p \ (1-p)^k \ p \cdot (1-p)^{n-k}.$$

$$= p^2 \ (1-p)^n .$$

$$P(X+Y=n) \overset{\text{TLP}}{=} \sum_{k=0}^{\infty} P(X=k,\ X+Y=n)$$

$$P(A) = \sum_i P(E_i \cap A)$$

$$E_i = (X=i)$$

$$= \sum_{k=0}^{\cancel{\infty}\ n} P(X=k,\ Y=n-k)$$

$$\overset{\text{indp.}}{=} \sum_{k=0}^{n} P(X=k)\, P(Y=n-k)$$

$$= \sum_{k=0}^{n} p\cdot(1-p)^k\, p\,(1-p)^{n-k}.$$

$$= \sum_{k=0}^{n} p^2 (1-p)^n.$$

$$= p^2 \cdot (1-p)^n \sum_{k=0}^{n} 1$$

$$= (n+1)\, p^2 (1-p)^n.$$

$$P(X=k \mid X+Y=n) = \frac{P(X=k,\ X+Y=n)}{P(X+Y=n)}$$

$$= \frac{p^2(1-p)^n}{(n+1)\, p^2 (1-p)^n}.$$

$$= \frac{1}{n+1}$$

Problem 4

[Midterm Jan'16]

Assignment 10: Temp. readings in a const. temp. medium are normally distributed with mean μ and std. dev. $\sigma = 0.2°$. The actual temp is estimated as

$$\bar{X} = \frac{1}{n}(X_1 + \dots + X_n)$$

a) $\mathbb{E}(\bar{X})$ & $Var(\bar{X})$

$$\mathbb{E}(\bar{X}) = \mathbb{E}\left(\frac{1}{n}(X_1 + \dots + X_n)\right)$$

$$X_i \overset{i.i.d}{\sim} N(\mu, \sigma) \quad : \sigma = 0.2$$

$$= \frac{1}{n}\mathbb{E}(X_1 + \dots + X_n)$$

$$= \frac{1}{n} \mathbb{E}X_1 + \dots + \mathbb{E}X_n$$

$$= \frac{1}{n}[\mu + \dots + \mu]$$

$$= \frac{1}{n} \cdot n\mu = \mu.$$

b) $$Var(\bar{X}) = \mathbb{E}(\bar{X}^2) - \left(\mathbb{E}(\bar{X})\right)^2 \rightarrow \mu^2.$$

$$\mathbb{E}(\bar{X}^2) = \mathbb{E}\left(\frac{1}{n^2}(X_1 + \dots + X_n)^2\right)$$

$$= \frac{1}{n^2}\mathbb{E}\left[(X_1 + \dots + X_n)^2\right]$$

$$= \frac{1}{n^2}\mathbb{E}\left[\sum_{i=1}^{n} X_i^2 + \sum_{\substack{i,j \\ i \neq j}} X_i X_j\right]$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\mathbb{E}(X_i^2) + \sum_{\substack{i,j \\ i \neq j}}\mathbb{E}(X_i X_j)$$

$$\mathbb{E}(X_i^2) = Var(X_i) + (\mathbb{E}(X_i))^2 = \sigma^2 + \mu^2.$$

$i \neq j \quad \mathbb{E}(X_i X_j) = \mathbb{E}(X_i)\, \mathbb{E}(X_j) = \mu \cdot \mu = \mu^2.$

$$\mathbb{E}(\bar{X}^2) = \frac{1}{n^2} \sum_{i=1}^{n} (\sigma^2 + \mu^2) + \sum_{\substack{i,j \\ i \neq j}} \mu^2.$$

$$= \frac{1}{n^2} \Big[n(\sigma^2+\mu^2) + \mu^2 \underbrace{\sum_{\substack{i,j \\ i \neq j}} 1}_{n^2 - n \text{ terms}} \Big]$$

$$= \frac{1}{n^2} \left[n(\sigma^2 + \cancel{\mu^2}) + (n^2 - \cancel{n}) \mu^2 \right]$$

$$= \frac{\sigma^2}{n} + \mu^2.$$

$$\mathrm{Var}(\bar{X}) = \mathbb{E}(\bar{X}^2) - (\mathbb{E}(X))^2$$

$$= \frac{\sigma^2}{n} + \mu^2 - (\mu)^2 = \frac{\sigma^2}{n}.$$

b) What value of n, the diff between $\bar{X}$ and μ is less than 0.1° with prob. atleast 0.95?

$$P(|\bar{X} - \mu| < 0.1^\circ) \overset{=}{\geq} 0.95$$

$$\bar{X} \sim N\left(\mu, \sqrt{\frac{\sigma^2}{n}}\right)$$

$$P\left(\left| \overbrace{\frac{\bar{X} - \mu}{\sqrt{\sigma^2/n}}}^{Z} \right| < \frac{0.1}{\sqrt{\sigma^2/n}} \right) = 0.95$$

$$P\left(|Z| < \frac{0.1\sqrt{n}}{\sigma}\right) = 0.95 \qquad \sigma = 0.2.$$

$$P\left(|Z| < \frac{\sqrt{n}}{2}\right) = 0.95$$

From Assignment

$$2\phi\left(\frac{\sqrt{n}}{2}\right) - 1 = 0.95$$

$$\phi\left(\frac{\sqrt{n}}{2}\right) = \frac{1+0.95}{2}.$$

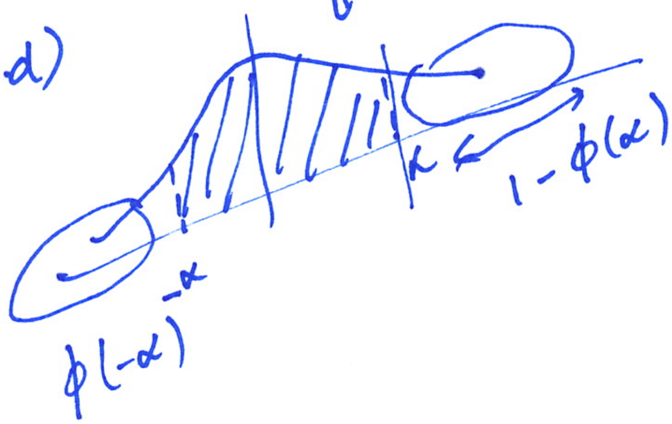
$$\frac{\sqrt{n}}{2} = \underbrace{\phi^{-1}}_{\text{qnorm}}\left(\frac{1+0.95}{2}\right).$$

$$n \approx 16.$$

$Z \sim N(0,1)$

$P(|Z| < \alpha)$
$= 2\phi(\alpha) - 1$

(4d)



Problem 5

Failure rate:

$$\lambda(x) = \frac{3}{x+1}$$

x: measured in months.

X: time to failure.

a) $\lambda(x) \downarrow x$

b) Find the distribution func?

$$F(x) = 1 - e^{-\int_0^x \lambda(t)\,dt}$$ ← Remember.

$$= 1 - e^{-\int_0^x \frac{3}{t+1} \cdot dt}$$

$$= 1 - e^{-3(\ln t+1)|_0^x}$$

$$= 1 - e^{-3\ln(x+1)}$$

$$= 1 - \frac{1}{(x+1)^3}$$

c) What is the prob that the system will survive another month after reaching median life?

→ prob that system will fail one month after median.

$$P(X \leq m+1 \,/\, X \geq m)$$ m: median.

$$= \frac{P(X \leq m+1,\ X \geq m)}{P(X \geq m)}.$$

$$= \frac{P(m \le X \le m+1)}{P(X \ge m)}$$

m: median value.

$$F(m) = \frac{1}{2}.$$

$$1 - \frac{1}{(x+1)^3} = \frac{1}{2}$$

$$\frac{1}{2} = \frac{1}{(x+1)^3}$$

$$x = \sqrt[3]{2} - 1$$

$$P(X \ge m) = 1 - Pr(X \le m) = \frac{1}{2}$$

$$Pr(m \le X \le m+1)$$

$$= F(m+1) - F(m)$$

$$= F(\sqrt[3]{2} - 1 + 1) - \frac{1}{2}$$

$$= F(\sqrt[3]{2}) - \frac{1}{2}$$

$$P(X \le m+1 \mid X \ge m) = \frac{F(\sqrt[3]{2}) - \frac{1}{2}}{\frac{1}{2}}$$

2.12) $f(x,y) = \begin{cases} c(x+y) & 0<x<2 \\ & 0<y<2 \\ 0 & \text{else.} \end{cases}$

a) What is the value of c ?

p.d.f integrates to 1.

$$\int_{x=0}^{2}\int_{y=0}^{2} f(x,y)\,dx\,dy = 1.$$

$$c\int_{x=0}^{2}\int_{y=0}^{2} (x+y)\,dx\,dy = 1.$$

$$c\int_{x=0}^{2} \left[xy + \frac{y^2}{2}\right]_0^2 dx = 1$$

$$c\int_{x=0}^{2} (2x+2)\,dx = 1.$$

$$2\cdot c\int_{x=0}^{2} (x+1)\,dx = 1$$

$$2c\left[\frac{x^2}{2} + x\right]_0^2 = 1.$$

$$2c\,[2+2] = 1$$

$$c = \frac{1}{8}$$

b) $\mathbb{E}(Y/X=x)$? (Best Linear Predictor).

$$\mathbb{E}(Y/X=x) = \int_{y=0}^{2} y\, f_{Y/X}(y/x)$$

$$f_{Y/X}(y/x) = \frac{\cancel{f_{X,Y}(X=x, y)}}{\cancel{f_Y(y)}}.$$

$$= \frac{f_{XY}(X=x, y)}{f_X(x)}$$

$$f_{XY}(X=x, y) = c\,(x+y)$$

$$f_X(x) = \int_{y=0}^{2} f_{XY}(x, y)\, dy.$$

$$= \int_{y=0}^{2} c\cdot(x+y)\, dy.$$

$$= c\left[xy + \frac{y^2}{2}\right]_0^2$$

$$= c\,[2x+2].$$

$$f_{Y/X}(y/x) = \frac{\cancel{c}(x+y)}{\cancel{c}(2x+2)}$$

Ex: Find $\mathbb{E}(Y/X)$