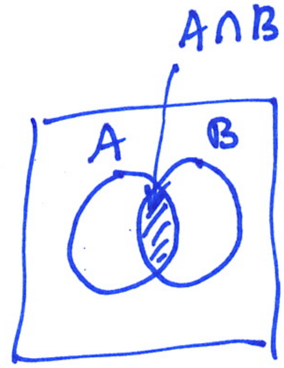
Tutorial

1.

A, B : Events

$P(A) = \frac{3}{4} \qquad P(B) = \frac{1}{3}$

S.T $\quad \frac{1}{12} \le P(A \cap B) \le \frac{1}{3}$



- $A \cap B \subset B$

$$P(A \cap B) \le P(B) = \frac{1}{3}$$

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$0 \le P(A \cup B) \le 1$$

$$P(A) + P(B) - P(A \cap B) \le 1$$

$$\frac{3}{4} + \frac{1}{3} - P(A \cap B) \le 1$$

$$P(A \cap B) \ge \frac{1}{12}$$

2. (Union Bound)

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

Induction Argument.

Step 1: Prove for $n = 1$.

$$P(A_1) \leq P(A_1)$$

$n = 2$.

S.T $P(A_1 \cup A_2) \leq P(A_1) + P(A_2)$

Proof: $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$ ①

$$0 \leq P(A_1 \cap A_2) \leq 1.$$

②

① + ② $\quad P(A_1 \cup A_2) \leq P(A_1) + P(A_2)$

Step 2: Assume that the hypothesis is true for $n = n-1$ and prove for $n = n$.

By our assumptions

Assume: $P\left(\bigcup_{i=1}^{n-1} A_i\right) \leq \sum_{i=1}^{n-1} P(A_i)$.

S.T $\quad P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$

LHS: $\quad P\left(\bigcup_{i=1}^{n} A_i\right) = P\left(\underbrace{\bigcup_{i=1}^{n-1} A_i}_{B} \cup A_n\right)$

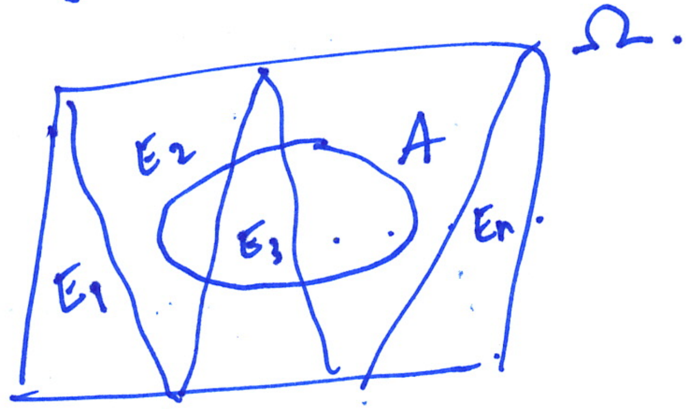
$= P(B \cup A_n) \leq P(B) + P(A_n)$.

$= P\left(\bigcup_{i=1}^{n-1} A_i\right) + P(A_n) \leq \sum_{i=1}^{n-1} P(A_i) + P(A_n$

$$= \sum_{i=1}^{n} P(A_i)$$

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

3. Total Probablity Law (TPL)



P(A) ?

Condition 1: E_i's are mutually exclusive.

$$E_i \cap E_j = \phi \quad i \neq j$$

$$E_i \cap E_j = \phi$$ ← Null set.

Condition 2: E_i's are exhaustive.

$$E_1 \cup E_2 \cdots \cup E_n = \Omega.$$

Then $$P(A) = \sum_{i=1}^{n} P(A \cap E_i)$$

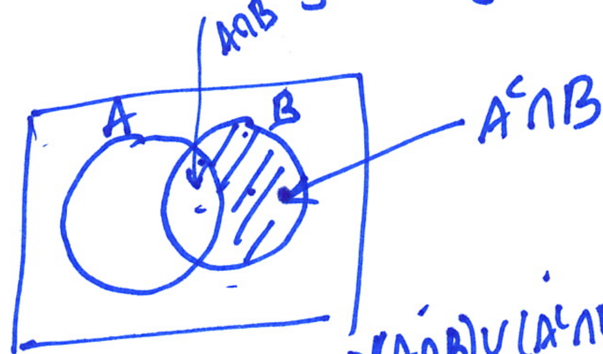
Correction from class:

This is known as the "Law of Total Probablity"

(1.1) $P(A) = \frac{1}{3} = P(B)$, $P(A \cap B) = \frac{1}{10}$ } Given.

a) $P(B^c) = 1 - P(B) = 1 - \frac{1}{3} = \frac{2}{3}$.

c) $P(A^c \cap B)$



$$P(B) = P((A \cap B) \cup (A^c \cap B)) = P(A \cap B) + P(A^c \cap B)$$

$$P(A^c \cap B) = P(B) - P(A \cap B) = \frac{1}{3} - \frac{1}{10} = \frac{7}{30}.$$

(Without Picture).

Proof: $P(B) \overset{TLP}{=} P(A \cap B) + P(A^c \cap B)$

TLP $E_1 = A$. $E_1 \cap E_2 = \phi$

$E_2 = A^c$ $E_1 \cup E_2 = \Omega$.

$$\frac{1}{3} = \frac{1}{10} + P(A^c \cap B).$$

$$P(A^c \cap B) = \frac{1}{3} - \frac{1}{10}.$$

d) $P(A^c \cup B)$

(b) is same as (d)

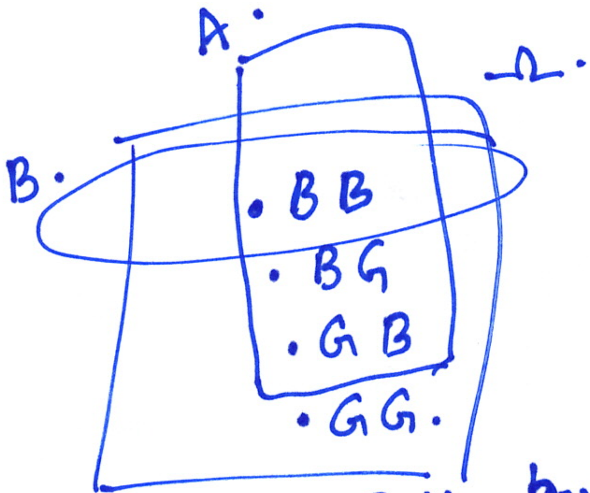
$$= P(A^c) + P(B) - P(A^c \cap B).$$

$$= 1 - P(A) + P(B) - P(A^c \cap B)$$

$$= 1 - \frac{1}{3} + \frac{1}{3} - \frac{7}{30}.$$

6).

Given:
All are equally likely.



A: One of the child is a boy.

B: Both boys.

$f(x) = kx + b$

STAT 321

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

(Def of cond. prob.)

$$= \frac{P(B)}{P(A)} = \frac{1/4}{3/4}.$$

$$= \frac{1}{3}.$$

1.2) A: Win First Race.

B: Win Second Race

Given: $P(A) = \cancel{0.6}$ 0.7 $\quad P(A \cap B) = 0.5$

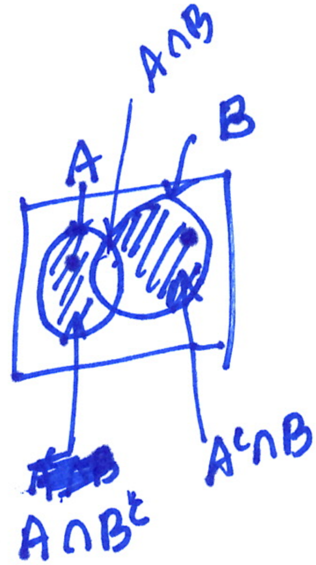
$P(B) = 0.6$

a) P(Wins atleast one race) $\to A \cup B$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$
$$= 0.8.$$

b) P(Wins exactly one race)

$\to (A^c \cap B) \cup (A \cap B^c)$



$$P((A^c \cap B) \cup (A \cap B^c))$$
$$= P(A^c \cap B) + P(A \cap B^c)$$

$$P(B) \overset{TLP.}{=} P(A \cap B) + P(A^c \cap B)$$
$$0.6 = 0.5 + P(A^c \cap B)$$
$$P(A^c \cap B) = 0.1$$

$$P(A) \overset{TLP}{=} P(A \cap B) + P(A \cap B^c)$$
$$0.7 = 0.5 + P(A \cap B^c)$$
$$P(A \cap B^c) = 0.2.$$

(Easy Way).

$$A \cup B = (A \cap B^c) \cup (A \cap B) \cup (A^c \cap B)$$
$$P(A \cup B) = P(A \cap B^c) + P(A \cap B) + P(A^c \cap B)$$

(c) P(Wins Neither Race)

$$= P((A \cup B)^c)$$

$$= 1 - P(A \cup B) = 1 - \overset{\text{(from (a))}}{0.8} = 0.2.$$

d) P(Wins Second race | Lost First)

$$= P(B \mid A^c)$$

$$\underset{\text{(Def of cond. prob)}}{=} \frac{P(A^c \cap B)}{P(A^c)} = \frac{\overset{\text{(from (b))}}{0.1}}{1 - P(A)} = \frac{0.1}{0.3} = \frac{1}{3}$$

(1.7) (Birthday Problem)

Given: n people in room

P(Atleast 2 of them share birthdate).

= 1 − P(None of the n people share birthdate).

$$1 \quad 2 \quad 3 \quad \cdots \quad n.$$

Total ways birthdays could occur.

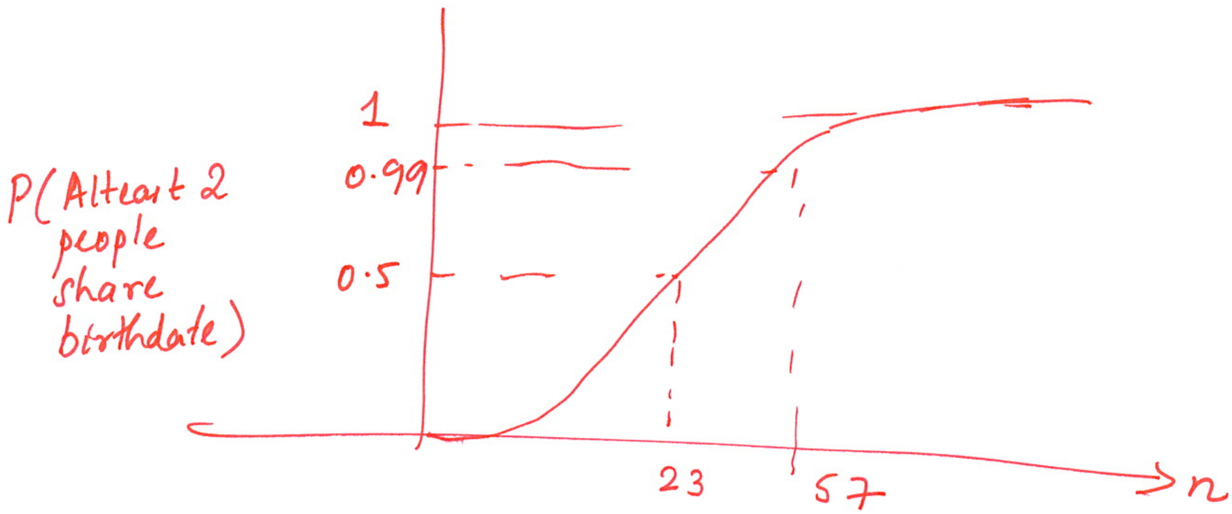
$$365 \quad 365 \quad \cdots \quad 365 = 365^n.$$

Possiblities that do not share birthdate.

$$365 \quad 364 \quad 363 \quad \cdots \quad 365-(n-1) =$$

$$= 365 \cdots (365-(n-1))$$

$$P(\text{None share birthdate}) = \frac{365!}{365^n (365-n)!}$$



[Corrected from class]