Reminder: Assignment 1 due on Friday

Recall: Moment Generating Function (MGF)

$$M(t) = \mathbb{E}(e^{tX}).$$

Problem 1: M.G.F of Binomial r.v

p: Probablity of Success $\quad q = 1-p.$

n: Number of trials.

$$P(X=k) = \binom{n}{k} p^k q^{n-k}$$

$$\begin{aligned} M(t) &= \mathbb{E}(e^{tX}) \\ &= \sum_{k=0}^{n} e^{tk} P(X=k) \\ &= \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k q^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} (e^t p)^k q^{n-k}. \\ &= (e^t p + q)^n. \quad \boxtimes \end{aligned}$$

$$\boxed{\begin{aligned} &(a+b)^n \\ &= a^n + \binom{n}{1} a^{n-1} b + \\ &\quad \cdots + b^n \end{aligned}}$$

Mean of random variable : $\mathbb{E}(X) = M'(0) = \frac{d}{dt} M(t)\Big|_{t=0}$

$$\begin{aligned} \mathbb{E}(X) &= \frac{d}{dt} (e^t p + q)^n \Big|_{t=0}. \\ &= n (e^t p + q)^{n-1} e^t p \Big|_{t=0}. \\ &= n (\underbrace{p+q}_{1})^{n-1} p. \\ &= np. \end{aligned}$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = \frac{d^2}{dt^2} M(t) \Big|_{t=0}$$

$$= \frac{d}{dt} np\, e^t (e^t p + q)^{n-1} \Big|_{t=0}$$

$$= np \left[e^t (e^t p + q)^{n-1} + e^t (n-1)(e^t p + q)^{n-2} (e^t p) \right] \Big|_{t=0}$$

$$= np \left[1\, (\underbrace{p+q}_{1})^{n-1} + 1 \cdot (n-1)(\underbrace{p+q}_{1})^{n-2}\, p \right].$$

$$= np + n(n-1)p^2.$$

$$Var(X) = np + n(n-1)p^2 - (np)^2.$$

$$= np - np^2 = np[\underbrace{1-p}_{q}]$$

$$= \underline{npq}.$$

<u>Problem 2</u>: [M.G.F of Geometric Distribution].

X follows a Geometric distribution if

$$\cancel{p(X=n)} = a^{n-1}(1-a) \qquad n = 1, 2, \dots, \infty$$

$P(X = n)$

a: parameter

$0 < a < 1$

$$M(t) = E[e^{tX}].$$

$$= \sum_{n=1}^{\infty} e^{tn} P(X=n)$$

$$= \sum_{n=1}^{\infty} e^{tn} a^{n-1} (1-a)$$

$$= \frac{1-a}{a} \sum_{n=1}^{\infty} e^{tn} a^n$$

$$= \frac{1-a}{a} \sum_{n=1}^{\infty} (a e^t)^n$$

$$= \frac{1-a}{a} \cdot \frac{a e^t}{1 - a e^t}$$

$$= \frac{(1-a) e^t}{1 - a e^t} \quad \boxtimes$$

Geometric ~~Ser~~ Series Formula. $|a| < 1$

$$\sum_{i=0}^{\infty} a^i = \frac{1}{1-a}$$

$$\sum_{i=1}^{\infty} a^i = \frac{a}{1-a}$$

Problem 4: Midterm Jan'16

Suppose we flip a coin until we obtain a Head. $P(\text{Head}) = 0.3$
$P(\text{Tail}) = 0.7$

a) What is the probability that we get the first head in the n^{th} flip? $n = 1, 2, \dots \infty$
(2 m).

$p = 0.3$
$q = 0.7$

$P(X = n)$

$X = \#$ of flips till first head.

1	2	3	....	n-1	n.
Tail	Tail	Tail	...	Tail	Head.
q	q	q	....	q	$(1-q) = p$

$$P(X=n) = \underbrace{q \times q \times \dots \times q}_{(n-1) \text{ times}} \times (1-q)$$

$$\cancel{= q^n (1-q)}$$

$$= q^{n-1} (1-q)$$

Geometric distribution with parameter q.

b) What is the probability that we flip atleast 3 times?

$$P(X \geq 3) = 1 - P(X < 3)$$
$$= 1 - P(X = 1) - P(X = 2)$$
$$= 1 - q^0 (1-q) - q(1-q)$$
$$= q^2$$

c) What is the expected number and variance of flips?

$$M \cdot G \cdot F = \frac{(1-a) e^t}{1 - a e^t}. \qquad a = q.$$
$$= \frac{(1-q) e^t}{1 - q e^t}.$$

$$E(X) = \frac{d}{dt} M(t) \Big|_{t=0}.$$
$$= \frac{d}{dt} \frac{(1-q) e^t}{1 - q e^t} \Big|_{t=0}$$
$$= (1-q) \frac{\left[(1 - q e^t) e^t - e^t (-q e^t) \right]}{(1 - q e^t)^2.} \Bigg|_{t=0}$$
$$= \cancel{(1-q)} \frac{\left[1 - \cancel{q} - \cancel{(-q)} \right]}{(1-q)^{\cancel{2}}}$$
$$= \frac{1}{1-q}$$

$$E(X^2) = \frac{d}{dt^2} M(t) \Big|_{t=0}$$

$$Var(X) = E(X^2) - [E(X)]^2$$

Problem 5:

[Tutorial 2.9]

[14 in Assignment 1].

Suppose $X_1, X_2, \ldots, X_n$ are independent random variables with distribution F.

$$U = \max_{1 \le i \le n} X_i$$

Find the distribution of U?

$n=3$

Eg:

	X_1	X_2	X_3	U
$P_1 \rightarrow$	1	2	1	2
	3	3	3	3
$P_2 \rightarrow$	2	2	1	2

≤ 3

$P_1 + P_2$.

Proof: $U = \max_{1 \le i \le n} X_i$

$$? \; F_U(u) = P(U \le u)$$

$$= P\left(\max_{1 \le i \le n} X_i \le u\right)$$

$$= P(X_1 \le u, X_2 \le u, \ldots, X_n \le u).$$

$$\overset{\text{independence}}{=} P(X_1 \le u)\, P(X_2 \le u) \ldots P(X_n \le u)$$

$$X_i \sim F_X$$

$$= F_X(u) \quad F_X(u) \quad \ldots \quad F_X(u)$$

$$= (F_X(u))^n$$

Eg: F : $P(X = i) = \frac{1}{m}$, $i = 1, \dots, m.$

c.d.f $F_X(i) = P(X \le i) = \sum_{j=1}^{i} \frac{1}{m} = \frac{i}{m}.$

$n = 5$:

$$F_u(u) = (F_X(u))^5$$

$$= \left(\frac{u}{m}\right)^5$$

p.m.f $P(X = u) = \underbrace{F_u(u)}_{P(X \le u)} - \underbrace{F_u(u-1)}_{P(X \le u-1)}$

$$= \left(\frac{u}{m}\right)^5 - \left(\frac{u-1}{m}\right)^5.$$

Problem 6: [Lecture 3, Pg 51]. Suppose the probability of finding oil at a certain location ~~of~~ has probability $p = 0.1$

a) How many wells should we dig to find oil with prob. ≥ 0.95?

X = # of wells with oil.

Let us dig n wells.

$X \sim \text{Binomial}(n, p)$

$P(\underbrace{\text{atleast one well with}}_{X > 0} \text{ oil}) \ge 0.95$

$$1 - P(X=0) \geq 0.95$$

$$1 - (1-p)^n \geq 0.95$$

$$1 - 0.9^n \geq 0.95$$

$$0.05 \geq 0.9^n.$$

$$n \geq \frac{\ln 0.05}{\ln 0.9} \approx 28.5$$

b) Atleast 2 wells with oil with prob ≥ 0.95?

$$P(X \geq 2) \geq 0.95$$

$$1 - P(X=0) - P(X=1) \geq 0.95$$

$$1 - 0.9^n - \underline{n}\, 0.9^{n-1}\, 0.1 \geq 0.95$$

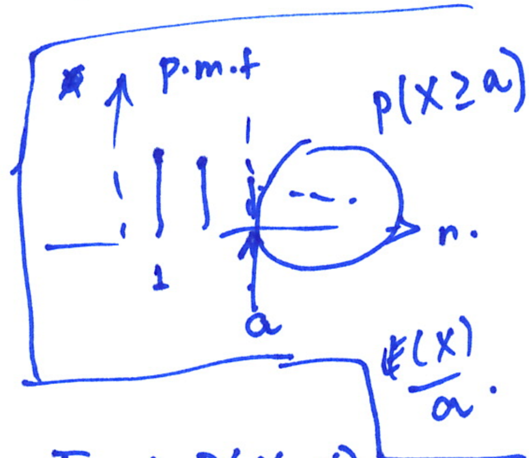
$$n \approx \underline{45}$$

Problem 7: [Markov Inequality] [Assignment 2].

If X is a positive random variable $X > 0$

For any number $a > 0$

$$P(X \geq a) \leq \frac{E(X)}{a}$$



Proof: $$E(X) = \sum_i i \cdot P(X=i)$$

$$= \sum_{i \geq a} i \cdot P(X=i) + \sum_{i<a} i\, P(X=i)$$

>0 $\quad$ >0

≥ 0

> 0

$$= \sum_{i \geq a} i \, P(X = i) + \cancel{\text{Qnty} > 0}$$

$$\geq \sum_{i \geq a} i \, P(X = i)$$

$i \geq a$

Replace i by a, $i \geq a$ so the term goes down

$$\geq \sum_{i \geq a} a \, P(X = i)$$

$$\geq a \sum_{i \geq a} P(X = i) = a \cdot P(X \geq a)$$

$$P(X \geq a) \leq \frac{E(X)}{a} \quad \blacksquare$$

Problem: 8 [Problem 11 in Ass 2]. Chebychev Inequality.

Let X be a random variable

$$P(|X - \underbrace{E(X)}_{\mu}| > r) \leq \frac{Var(X)}{r^2}$$

$P(|X - E(X)| > r)$

Proof: $\operatorname{Var}(X) = \mathbb{E}(X - \underbrace{\mathbb{E}(X)}_{\mu})^2$ $\quad \mu$: mean.

$$= \sum_i \cancel{\mathbb{E}} (i-\mu)^2 \cdot P(X=i)$$

$$= \sum_{i:|i-\mu|\ge r} (i-\mu)^2 P(X=i) + \underbrace{\sum_{i:|i-\mu|<r} \underbrace{(i-\mu)^2}_{>0} \underbrace{P(X=i)}_{>0}}_{>0}$$

Drop this term!

$$\ge \sum_{i:|i-\mu|\ge r} \underbrace{(i-\mu)^2} P(X=i)$$

$$\ge \sum_{i:|i-\mu|\ge r} r^2 P(X=i)$$

$$\ge r^2 \underbrace{\sum_{i:|i-\mu|\ge r} P(X=i)}_{P(|X-\mu|\ge r)}$$

$$P(|X-\mu| \ge r) \le \frac{\operatorname{Var}(X)}{r^2}$$

Assignment 1.

4. b) Key idea : $\cancel{B_i}$ $A_i = B_i^c$ (Then use part (a)) (De-Morgan's Law)

5. b) P(Success Comm.)

$$\stackrel{T.L.P}{=} \sum_m P(\text{Succes. Comm}, \text{ m antenna's are down})$$

$$\stackrel{(\text{Cond})}{=} \sum_m P(\text{m antennas are down}) \underbrace{P(\text{Succes Comm} \mid \text{m antena's are down})}$$

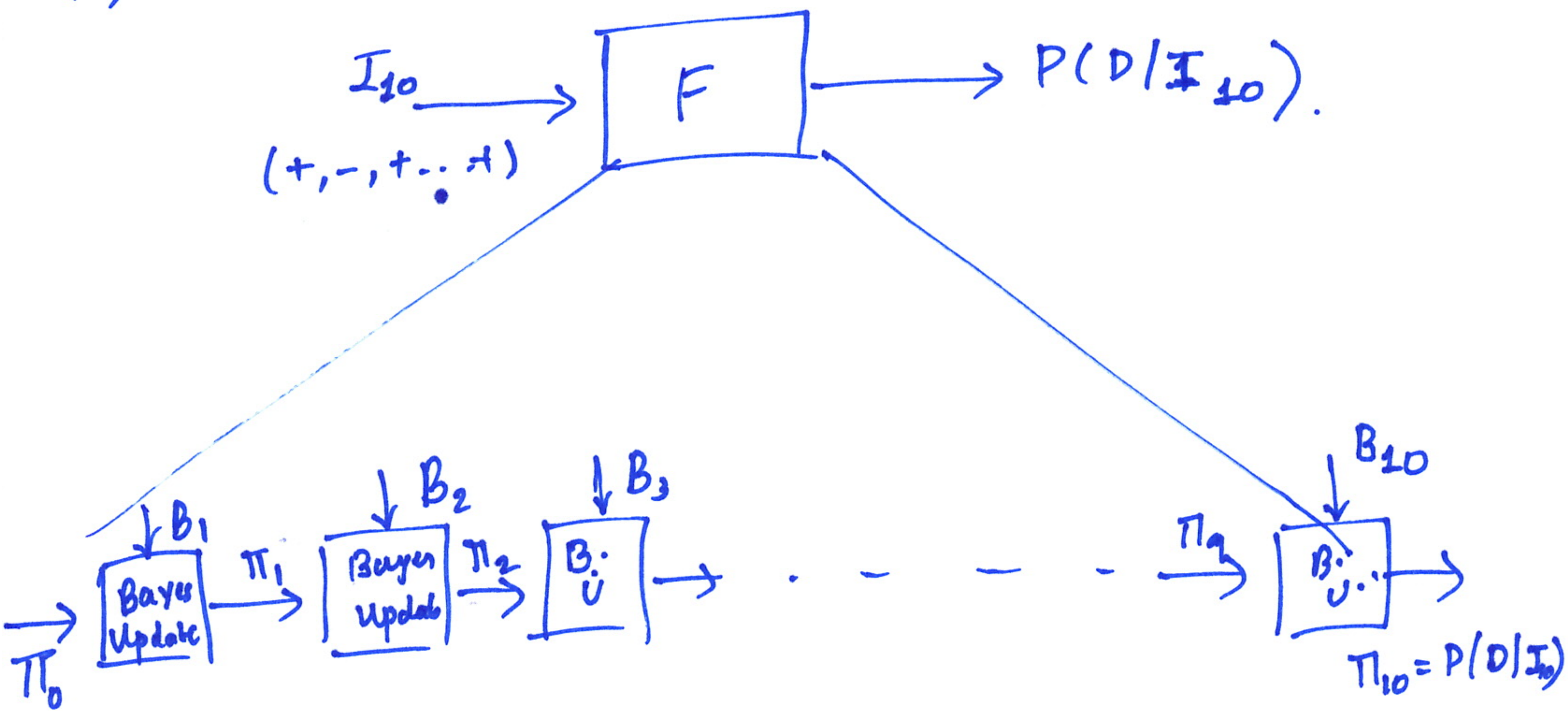
Values from Table

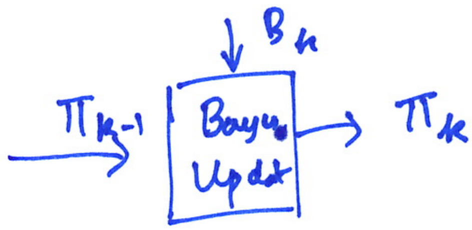
(a) $\Rightarrow$ # of orderings for succ. comm.

$$\frac{\text{\# of ordering for succ. comm.}}{\text{Total no: of orderings.}}$$

11. d) Bayes Update : 2 times.

12)





$B_k = +$

$$\pi_k = \frac{\pi_{k-1}\, P(B_k = + | D)}{\pi_{k-1}\, P(B_k = + | D) + (1-\pi_{k-1})\, P(B_k = + | ND)}.$$

$k = 1$. $$\pi_1 = \frac{\pi_0 \cdot 0.8}{\pi_0\, 0.8 + (1-\pi_0)\, 0.2}.$$

$\pi_0 = 0.15$

$I_{10} = (+, -, + \ldots, -)$.

$+ - 1$

$- - 0$

Defective

(First col. part of Table)

sample (c(1,0), 1, prob = (0.8, 0.21)) T_1

$\vdots$

sample (c(1,0), 1, prob = (0.75, 0.25)) T_{10}

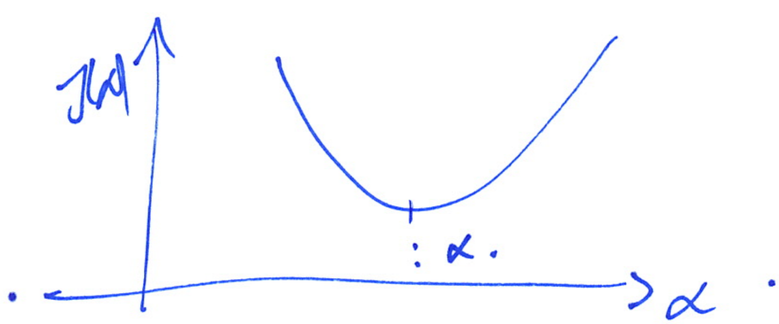
rbinom(10, 1, prob = (0.8, .. , 0.75))

Non-Def

(2nd col. of Table)

sample (c(1,0), 1, prob = (0.2, 0.8))

(5c)



From part (a)

I_{10} (i)

(ii)

(iii)

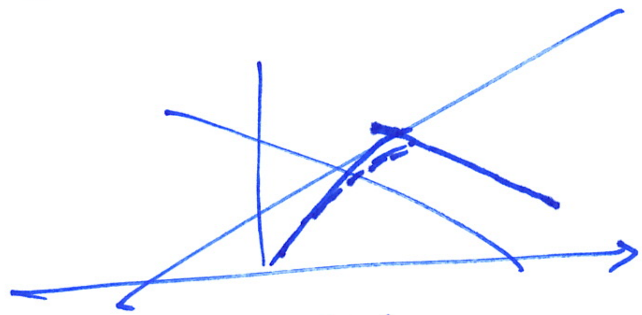
15. $f(x) = c \cdot x\,(1280 - x)^4 \quad x = 1, \dots, 1200.$

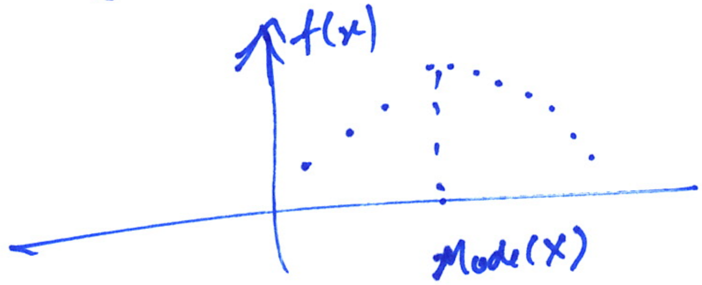
(b) $E(X) \to$ Find using R.

$\cancel{P(X \geq}$ $\quad P(X > \underbrace{E(X)}_{100})$

$$= \sum_{i=101}^{1200} f(i) \quad \leftarrow \text{Use R.}$$

(c) Mode(X)





$f(x) = f = [f(1) \;\dots\; f(1200)].$

which(max(f)==f) = Mode(X)

$$P(X > \underbrace{\text{Mode}(X)}_{120}) = \sum_{i=120}^{1200} f(i) \quad \leftarrow \text{Use R.}$$