ELEC 321

Today:

1. Review: Discrete & Continous r.v
2. Moment Generating Function
3. Binomial, Geometric, Poisson, Exponential
4. Failure rate
5. Change of variable.

Discrete random variable X

p.m.f $\quad f(x) = P(X = x)$

c.d.f $\quad F(x) = P(X \leq x) = \sum_{k \leq x} f(k)$

Expected value $\quad \mathbb{E}(X) = \sum_{x} x\, f(x)$

Eg

Binomial

Geometric

Poisson.

Continous random variable X

p.d.f $\quad f(x) \qquad P(a < X < b) = \int_a^b f(x)\,dx$

c.d.f $\quad F(x) = \int_{-\infty}^{x} f(x)\,dx$

$$= \int_{-\infty}^{b} f(x)\,dx - \int_{-\infty}^{a} f(x)\,dx$$

$$= F(b) - F(a)$$

Expected value $\quad \mathbb{E}(X) = \int x\, f(x)\,dx$

Eg: Exponential, Normal, Uniform

Variance: $\quad Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$

Assignment 1.1
Qns 3,5,6,8,9 were graded with 10pts each
For the following questions it is preferable to meet with the corresponding TA during OH

Qn. 5 Anup (Oct 13: 5-6pm)
Qn. 8 Mrinmoy (Oct 18: 5-6pm)
Qn. 9 Ali (Oct 11: 5-6pm)

For other questions, please email one of us.

Problem 1 [B.2]

$$F(x) = \begin{cases} 1 - \frac{1}{x^2} & x > 1 \\ 0 & \text{else} \end{cases}$$

a) $P(1.5 < X < 2.5)$

b) $P(X \le 3 \mid X > 2)$

c) $E(X)$ & $Var(X)$

d) Median (X)

a) $P(1.5 < X < 2.5) = F(2.5) - F(1.5)$

$$= \left(1 - \frac{1}{2.5^2}\right) - \left(1 - \frac{1}{1.5^2}\right)$$

b) $P(X \le 3 \mid X > 2)$

$$= \frac{P(X \le 3, X > 2)}{P(X > 2)}$$

$$= \frac{P(2 < X \le 3)}{P(X > 2)} = \frac{F(3) - F(2)}{1 - F(2)}$$

$$= \frac{(1 - 1/3^2) - (1 - 1/2^2)}{1 - (1 - 1/2^2)}$$

$P(X > 2)$
$= 1 - P(X \le 2)$
$= 1 - F(2)$

c) $E(X)$ $\quad f(x) = \frac{d}{dx} F(x)$

$$= \begin{cases} 2/x^3 & x > 1 \\ 0 & \text{else.} \end{cases}$$

$$E(X) = \int x \; f(x) \, dx$$

$$= \int_1^{\infty} x \cdot 2/x^3 \, dx.$$

$$= \int_1^{\infty} 2/x^2 \, dx$$

$$= 2 \left[-\frac{1}{x} \right]_1^{\infty} = 2 //$$

$$E(X^2) = \int x^2 \; f(x) \, dx$$

$$= \int_1^{\infty} x^2 \cdot 2/x^3 \, dx$$

$$= \int_1^{\infty} 2/x \, dx$$

$$= 2 \ln x \Big]_1^{\infty} = \infty$$

$$Var(X) = E(X^2) - (E(X))^2 = \infty.$$

d) Median : $F_X(m) = 0.5$

$$1 - \frac{1}{m^2} = \frac{1}{2}$$

$$m = \sqrt{2} //$$

Moment Generating Function. (M.G.F)

$$M_X(t) = \mathbb{E}[e^{tX}]$$

$$\mathbb{E}(X) = \frac{d}{dt} M(t) \Big|_{t=0}$$

$$\mathbb{E}(X^2) = \frac{d^2}{dt^2} M(t) \Big|_{t=0}$$

Problem 2 M.G.F of Binomial r.v.

$X \sim Bin(n,p)$ $P(X = k) = \binom{n}{k} p^k q^{n-k}$: $q = 1-p$

Side Note:

$$\left[\begin{array}{l} X \sim Bin(n,p) \\ X = \sum_{i=1}^{n} Y_i \qquad Y_i \sim Bernoulli(p) \end{array} \right]$$

$$M_X(t) = \mathbb{E}(e^{tX})$$

$$= \sum_{k=0}^{n} P(X=k)\, e^{tk}.$$

$$= \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} e^{tk}.$$

$$= \sum_{k=0}^{n} \binom{n}{k} (e^t p)^k q^{n-k}.$$

Binomial expansion

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

$$= (e^t p + q)^n$$

For ex: M.G.F $(e^t p + q)^{2n} \sim Bin(2n, p)$

$X \sim Bin(n, p)$

$$M_X(t) = (e^t p + q)^n \quad : q = 1-p.$$

$$\mathbb{E}(X) = \frac{d}{dt} M_X(t) \Big|_{t=0}.$$

$$= \frac{d}{dt} (e^t p + q)^n \Big|_{t=0}.$$

$$= n (e^t p + q)^{n-1} (e^t p) \Big|_{t=0}$$

$$= n (e^0 p + q)^{n-1} (e^0 p)$$

$$= np (\underbrace{p + q}_{1})^{n-1}$$

$$= \underline{\underline{np}}$$

$$\mathbb{E}(X^2) = \frac{d^2}{dt^2} M(t) \Big|_{t=0}$$

} Exercise

$$= np + n(n-1)p^2$$

$$Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 \text{ [Exercise]}.$$

$$= npq$$

Geometric Distribution.

Problem 3 [Winter '16] Suppose we flip a coin until we obtain a head with $P(\text{Head}) = p$

$$P(\text{Tail}) = q = 1-p$$

a) What is the probability that we get a first head in n^{th} trial?

b) What is the M.G.F ?

c) Expected number & variance of flips until first head ?

a) $X =$ # flips until first head

$P(X = n)$

$$\begin{array}{ccccc} 1 & 2 & \cdots & n-1 & n \\ \text{Tails} & \text{Tails} & \cdots & \text{Tails} & \text{Head.} \\ \underbrace{q \quad\quad q} & \underbrace{\cdots\cdots} & \underbrace{\quad q}_{n-1} & & \underbrace{p.}_{1} \end{array}$$

$$P(X = n) = q^{n-1} p = q^{n-1}(1-q).$$

b) M.G.F $\quad M(t) = \mathbb{E}[e^{tX}].$

$$= \sum_{n=1}^{\infty} P(X = n)\, e^{tn}.$$

$$= \sum_{n=1}^{\infty} q^{n-1}(1-q)\, e^{tn}$$

$$= \frac{1-q}{q} \sum_{n=1}^{\infty} (q e^{t})^{n}$$

Geometric Sum $\displaystyle\sum_{k=1}^{\infty} a^k = \frac{a}{1-a} \;:\; |a|<1$

$$= \frac{1-q}{q} \cdot \frac{qe^t}{1-qe^t} \quad [\text{Using } a = qe^t]$$

$$= \frac{(1-q)\,e^t}{1-qe^t}$$

c) Expectation:

$$\mathbb{E}(X) = \frac{d}{dt} M(t)\Big|_{t=0}$$

$$= \frac{d}{dt} \frac{(1-q)\,e^t}{1-qe^t}\Big|_{t=0}.$$

Use quotient rule for differentiation

$$= (1-q) \left[\frac{(1-qe^t)\,e^t - e^t(-qe^t)}{(1-qe^t)^2}\right]\Bigg|_{t=0}$$

$$= (1-q) \left[\frac{(1-q) - (-q)}{(1-q)^2}\right]$$

$$= \underline{\underline{\frac{1}{1-q}}}$$

Exercise: Find $\mathbb{E}(X^2)$

$$= \frac{d^2}{dt^2} M_X(t)\Big|_{t=0}$$

[Quotient rule].

$Var(X)$?

Poisson & Exponential R.V.

Poisson : (Discrete R.V)

$X \sim \text{Poisson}(\lambda)$

$P(X = k) = e^{-\lambda} \dfrac{\lambda^k}{k!}$

$\mathbb{E}(X) = \lambda$

$\text{Var}(X) = \lambda$

Exponential : (Continous R.V)

$X \sim \text{Exp}(\lambda)$

$f_X(x) = \lambda e^{-\lambda x} \qquad x > 0$

$F_X(x) = 1 - e^{-\lambda x}$

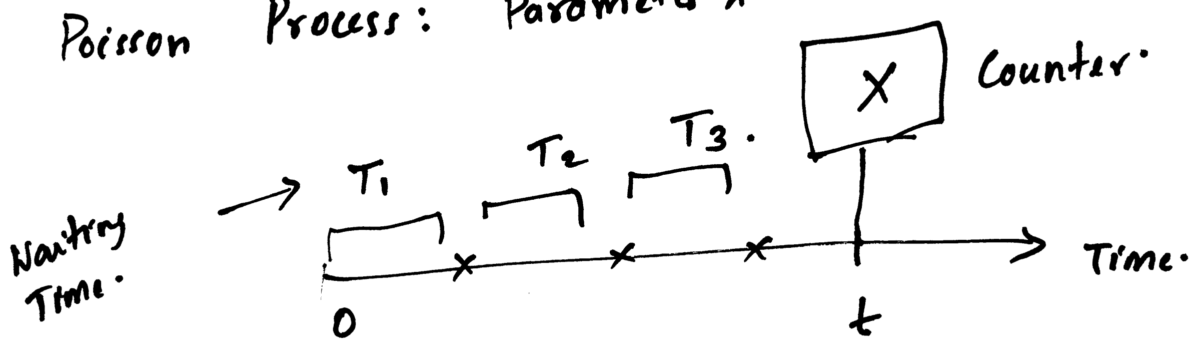
$P(X > h) = e^{-\lambda h}$

$\mathbb{E}(X) = 1/\lambda$

$\text{Var}(X) = 1/\lambda^2$

Memoryless Property: $P(X > t+h \mid X > t) = P(X > h)$
$= e^{-\lambda h}$

Poisson Process: Parameter λ



$X \sim \text{Poisson}(\lambda t) \qquad T_i \sim \text{Exp}(\lambda)$

Problem 4 A certain event occurs at a rate
[B·1] $\lambda = 3/hr$.

a) What is the probability of 5 occurrence in next 2 hrs?

b) Probability of waiting more than 30 minutes for next occurrence of A?

c) Expected wait time until ~~that~~ first occurrence of A? Variance?

d) Expected number of occurrence in next 10 mins?

a) X = # occurrence in next 2 hrs.

$$X \sim \text{Poisson}(\lambda t) = \text{Poisson}(3 \cdot 2)$$
$$= \text{Poisson}(6)$$

$$P(X = 5) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$
$$= e^{-6} \, 6^5 / 5!$$

b) T: First waiting time. $T \sim \exp(\lambda)$
$= \exp(3)$

$$P(T > \tfrac{1}{2} \text{hr})$$
$$= e^{-\lambda h} = e^{-3 \times \frac{1}{2}} = e^{-1.5}$$

c) T: First waiting time.

$T \sim \exp(3)$

$$E(T) = \frac{1}{\lambda} = \frac{1}{3} \quad //$$

$$Var(T) = \frac{1}{\lambda^2} = \frac{1}{3^2} \quad //$$

d) Expected number of occurences in next 10 minutes

X = # of occurence in 10 minutes ($\frac{1}{6}$ hrs)

$$X \sim \text{Poisson}(\lambda t) = \text{Poisson}\left(3 \cdot \frac{1}{6}\right)$$
$$= \text{Poisson}\left(\frac{1}{2}\right)$$

$$E(X) = \lambda t = \frac{1}{2}$$

$X = 1$ w.p 0.5

$= 0$ w.p 0.5

Failure rate: [Related to exponential distribution]

Failure rate: $\lambda(t)$ of a system

X: Lifetime of a system.

$$F_X(x) = 1 - e^{-\int_0^x \lambda(t)\, dt}$$

$$\Rightarrow \quad \lambda(t) = -\frac{d}{dt} \ln(1 - F(t))$$

$\lambda(t)$ is constant: System doesn't wear out

$\lambda(t) \uparrow t$: System weakens with time

$\lambda(t) \downarrow t$: System improves with time.

Problem 7 [Winter 16]

Suppose the lifetime of a system, in years, has the distribution

$$F_X(x) = 1 - e^{-x^3}$$

a) Is the part wearing out, getting stronger or not changing with time?

$$\lambda(t) = -\frac{d}{dt} \ln(1 - F(t))$$

$$= -\frac{d}{dt} \ln(1 - (1 - e^{-t^3}))$$

$$= -\frac{d}{dt} \ln e^{-t^3}$$

$$= 3t^2$$

$\lambda(t) \uparrow t$

Part is wearing out.

b) Median life of system:

$$F(m) = 0.5$$

$$1 - e^{-m^3} = 0.5.$$

$$0.5 = e^{-m^3}$$

$$m = \sqrt[3]{\ln 2}$$

c) Calculate the probability that the part will survive 6 extra months after serving for 1.5 years?

X = Lifetime of a product.

$$P(X > 1.5 + 0.5 \mid X > 1.5)$$

$$= \frac{P(X > 2,\ X > 1.5)}{P(X > 1.5)}$$

$$= P(X > 2) / P(X > 1.5)$$

$$= \frac{1 - F(2)}{1 - F(1.5)} = \frac{e^{-2^3}}{e^{-1.5^3}}$$

Problem 5
Fall 2015

Suppose U is a uniform random variable in the interval $(0, 1)$
Let $X = e^U$

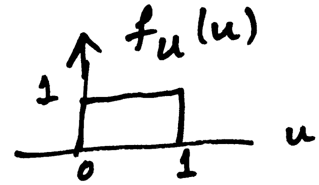
a) Range of X ?

b) Distribution & Density of X ?

c) $E(X)$ & $Var(X)$?

$f_u(u) = 1$

$F_u(u) = u.$



a) Range of X

$U \sim Unif(0, 1)$

e^U lie between 1 and e.

b) Find the distribution ?

c.d.f $\quad F_X(x) = P(X \le x)$

$$= P(e^U \le x)$$

$$= P(U \le \ln x)$$

$$= F_U(\ln x)$$

$$= \ln x$$

p.d.f $\quad f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx} \ln x = \frac{1}{x}.$

c) Exercise: $\quad E(X) \quad \& \quad Var(X) = (e-1)^2/2$?

$E(X) = \int^e x \, f_X(x)\, dx = (e-1)$

$E(X^2) = \int_1^e x^2 f_X(x)\, dx = (e^2-1)/2$

Problem 6 Suppose X has the distribution

[B.3]

$$F_X(x) = 1 - \frac{1}{x^2} \qquad x > 1$$

a) $Y = X^2$ $\qquad$ b) $V = \frac{1}{X}$

Distribution & density of Y and V. ?

a)

$$F_Y(y) = P(Y \le y)$$

$1 < X < \infty$

$1 < Y < \infty$

$$= P(X^2 \le y)$$

$$= P(X \le \sqrt{y})$$

$$= F_X(\sqrt{y})$$

$$= 1 - \frac{1}{(\sqrt{y})^2} = 1 - \frac{1}{y}$$

$$f_Y(y) = \frac{1}{y^2} \quad \text{(density)}$$

b)

$$1 < X < \infty$$

$$0 < V < 1.$$

$$F_V(v) = P(V \le v)$$

$$= P\left(\frac{1}{X} \le v\right)$$

$$= \cancel{P(X \ge v)} \; P\left(X \ge \frac{1}{v}\right)$$

$$= 1 - F_X\left(\frac{1}{v}\right)$$

$$= 1 - \left(1 - \frac{1}{(1/v)^2}\right) = v^2$$

$$f_V(v) = 2v$$