Tutorial 2

Problem 1.8 (set A)

Random variable X

x_i	1	2	3	...	10
$P(X=x_i)=f(x_i)$	α	$2\cdot\alpha$	$3\cdot\alpha$	...	$10\cdot\alpha$

↑ pmf

a) $\alpha = ?$; We know $\sum_{i=1}^{n} f(x_i) = \sum_{i=1}^{n} P(X=x_i) = 1$

$P(X=x_i) = \alpha i$ for $i=1,\dots,10$, $x_i = i$, $i=1,\dots,10$

$$1 = \sum_{i=1}^{n} P(X=x_i) = \sum_{i=1}^{10} P(X=i) = \sum_{i=1}^{10} \alpha i = \alpha \sum_{i=1}^{10} i = \alpha \frac{10(10+1)}{2} = 55\alpha$$

Sum of integers from 1 to n is $\frac{n(n+1)}{2}$ - can be proven by induction

$$55\alpha = 1 \Rightarrow \alpha = \frac{1}{55}$$

b) $EX = ?$; $E(X) = \sum_{i=1}^{n} x_i P(X=x_i)$

$P(X=x_i) = \frac{1}{55} i$, $x_i = i$ for $i=1,\dots,10$

$$E(X) = \sum_{i=1}^{n} x_i P(X=x_i) = \sum_{i=1}^{10} i \cdot \frac{1}{55} i = \frac{1}{55}\sum_{i=1}^{10} i^2 = \frac{1}{55} \frac{10\cdot(10+1)(20+1)}{6} = 7$$

Sum of squares of integers 1 to n is $\frac{n(n+1)(2n+1)}{6}$

c) $P(X \le 2 \mid X \le 5) = \dfrac{P((X\le 2)\cap(X\le 5))}{P(X\le 5)}$ → conditional probability formula

$$P((X\le 2)\cap(X\le 5)) = P(X\le 2)$$

$$P(X\le 2) = P(X=1) + P(X=2) = \frac{1}{55} + \frac{2}{55} = \frac{3}{55}$$

$$P(X\le 5) = P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5) =$$

$$= \frac{1}{55} + \frac{2}{55} + \frac{3}{55} + \frac{4}{55} + \frac{5}{55} = \frac{15}{55}$$

$$\therefore \quad P(X\le 2 \mid X\le 5) = \frac{P(X\le 2)}{P(X\le 5)} = \frac{3/55}{15/55} = \frac{1}{5}$$

Problem 1.9

10 keys → 1 correct , no replacement (i.e. after trying not putting the key back)

Define $A_i = \{i\text{-th attempt works}\}$ ○ ○ ● ... ○
1 2 3 10

a) $P(A_1) = ?$

$P(A_1) = P(\text{first attempt works}) = \frac{1}{10}$

b) $P(A_2) = P(\text{first did not work} \cap \text{second worked}) = \frac{\cancel{9}}{10} \cdot \frac{1}{\cancel{9}} = \frac{1}{10}$

$P(A_1^c \cap A_2) = P(A_2 | A_1^c) \cdot P(A_1^c)$, where

$P(A_1^c) = 1 - P(A_1) = \frac{9}{10}$; $P(A_2 | A_1^c) = \frac{1}{9}$ (one correct out of 9 remaining)

Another way: $P(A_2) = P(A_1^c \cap A_2) + \underbrace{P(A_1 \cap A_2)}_{=0}$ =0 because only 1 correct key

c) for $i = 1, \ldots, 10$

$P(A_i) = P(\text{1 up to } i-1 \text{ did not work} \cap i\text{th attempt worked}) =$

result from Assignment1 Problem 6 → $= P(A_1^c \cap A_2^c \cap \ldots \cap A_{i-1}^c \cap A_i) = P(A_i | A_1^c \cap A_2^c \cap \ldots \cap A_{i-1}^c) \times P(A_{i-1}^c | A_1^c \cap \ldots \cap A_{i-2}^c)$

$\ldots \times P(A_2^c | A_1^c) \cdot P(A_1^c) = \frac{1}{\cancel{n-i+1}} \times \frac{\cancel{n-i+1}}{\cancel{n-i+2}} \times \ldots \times \frac{\cancel{8}}{\cancel{9}} \times \frac{\cancel{9}}{10} = \frac{1}{10}$

d) Let X = # of attempts till first success $\mathbb{E}(X) = ?$

maximum # of attempts = 10

x_i	1	2	3	...	10
$f(x_i) = P(X = x_i)$	1/10	1/10	1/10	...	1/10

$$\mathbb{E}X = \sum_{i=1}^{n} f(x_i) \cdot x_i = \sum_{i=1}^{n} x_i \cdot P(X = x_i)$$

$x_i = i, \quad i = 1, \ldots, 10 \quad ; \quad f(x_i) = 1/10, \quad i = 1, \ldots, 10$

$$\mathbb{E}X = \sum_{i=1}^{10} \frac{1}{10} \cdot i = \frac{1}{10} \cdot \frac{10(10+1)}{2} = 5.5$$

Expected # of attempts till success = 5.5 ⇒ we expect success to occur on 6th (or 5th) try

Meaning of expected value: if we repeat the experiment many times, on average, success will happen on 5th or 6th attempt.

Problem 1.10

10 keys → 1 correct , yes replacement (we put the key back after trying)

Define A_i = {success on the i-th attempt}

a) $P(A_1) = ?$

$P(A_1) = 1/10$ (10 keys, 1 correct)

b) $P(A_2) = P(A_2 | A_1^c) \cdot P(A_1^c) = \frac{1}{10} \cdot \frac{9}{10}$

→ the # of keys did not change

c) for $i = 3, 4, \ldots$

$P(A_i) = \left(\frac{9}{10}\right)^{i-1} \cdot \frac{1}{10}$ (i-1) attempts failed and the i-th one worked

Formally $P(A_i) = P(A_i | A_1^c \cap \ldots \cap A_{i-1}^c) \times P(A_{i-1}^c | A_1^c \ldots A_{i-2}^c) \times \ldots \times P(A_2^c | A_1^c) \cdot P(A_1^c)$

result #6 from assignment 1

d) In theory, it could take up to ∞ number of tries if we keep replacing the key

Let X = # of attempts until first success, $\mathbb{E}(X) = ?$

x_i	1	2	3	...	i	...	∞
$f(x_i) = P(X = x_i)$	$1/10$	$9/10 \cdot 1/10$	$(9/10)^2 \cdot 1/10$	...	$(9/10)^{i-1} \cdot 1/10$	...	

$$\mathbb{E}X = \sum_{i=1}^{n} x_i \cdot P(X = x_i)$$

In this case $x_i = i, \quad i = 1, \ldots, \infty$

$$P(X = x_i) = \left(\frac{9}{10}\right)^{i-1} \cdot \frac{1}{10}$$

A

$$\mathbb{E}X = \sum_{i=1}^{\infty} x_i \cdot P(X = x_i) = \sum_{i=1}^{\infty} i\left[\left(\frac{9}{10}\right)^{i-1} \cdot \frac{1}{10}\right] = \sum_{i=1}^{\infty} i \left(\frac{9}{10}\right)^{i-1}\left(1 - \frac{9}{10}\right) =$$

$$= \sum_{i=1}^{\infty} i \cdot \left(\frac{9}{10}\right)^{i-1} - \sum_{i=1}^{\infty} i \cdot \left(\frac{9}{10}\right)^{i} = \sum_{k=0}^{\infty} (k+1)\left(\frac{9}{10}\right)^{k} - \sum_{k=0}^{\infty} k \left(\frac{9}{10}\right)^{k} =$$

(first addendum is 0)

$$= \sum_{k=0}^{\infty} \left(\frac{9}{10}\right)^{k} (k+1-k) = \sum_{k=0}^{\infty} \left(\frac{9}{10}\right)^{k} = \underbrace{\frac{1}{1 - \frac{9}{10}}}_{\text{sum of infinite geometric series}} = 10$$

Another proof: We know that $\sum_{i=0}^{\infty} p^i = \frac{1}{(1-p)}$ ← if $p < 1$

Differenciate both sides w.r.t. p $\sum_{i=0}^{\infty} i\, p^{i-1} = \frac{1}{(1-p)^2}$

Notice that $\sum_{i=0}^{\infty} i\, p^{i-1} = \sum_{i=1}^{\infty} i\, p^{i-1} = \frac{1}{(1-p)^2}$ Because first addendum is zero

Multiply both sides by $(1-p)$

$$\sum_{i=1}^{\infty} i p^{i-1}(1-p) = \frac{1}{1-p}$$

Now (A) is exactly the left side with $p = \frac{9}{10}$

Problem 5 assignment 1

n antenae, $m < n$ are broken

system works if there are no ≥ 2 broken antenae in sequence

Hint $n-m$ working antenae

$$\uparrow \_ \uparrow \_ \uparrow \_ \uparrow \_ \uparrow \dots \uparrow \_ \uparrow$$

$n-m$ working

places for broken antenae

# of places for broken antenae $n-m+1$

Things we need to know to solve tutorial p. 11

① Conditional probability for events A and B where $P(A), P(B) > 0$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad , \quad P(B|A) = \frac{P(B \cap A)}{P(A)}$$

$P(A \cap B) = P(B \cap A)$ intersection is transitive

① $P(A \cap B) = P(A|B) \cdot P(B) = P(B|A) \cdot P(A)$

② Law of total probability

for any events A and B, $P(A), P(B) > 0$

$$P(A) = P(A \cap B) + P(A \cap B^c) = P(A|B) \cdot P(B) + P(A|B^c) \cdot P(B^c) \quad ②$$

$$= P(B|A) \cdot P(A) + P(B^c|A) \cdot P(A)$$

③ For any events A and B, $P(A), P(B) > 0$

$P(B|A) + P(B^c|A) = 1$ $\qquad$ $P(A|B) + P(A^c|B) = 1$ ③

$P(A|B) + P(A|B^c) \neq 1$

Tutorial problem 1.11

When I write (1) => means using (1)

Given: $E = \{\text{patient sick}\}$

$T_i = \{\text{test result } i \text{ positive}\}$, $i = 1, 2, 3, 4, 5$

T_i are conditionally indep. given E and given E^c. It means:

$$P(T_1 \cap T_2 \cap T_3 \cap T_4 \cap T_5 \mid E) = P(T_1 \mid E) \cdot P(T_2 \mid E) \ldots P(T_5 \mid E)$$
$$P(T_1 \cap \ldots \cap T_5 \mid E^c) = P(T_1 \mid E^c) \cdot P(T_2 \mid E^c) \ldots P(T_5 \mid E^c)$$

(4) ← shorthand notation

Known: $P(E)$, $P(T_i \mid E)$, $P(T_i^c \mid E^c)$

$\hookrightarrow P(T_i^c \mid E) = 1 - P(T_i \mid E)$, $P(T_i^c \mid E^c) = 1 - P(T_i \mid E^c)$

$\Rightarrow P(T_i \mid E^c) = 1 - P(T_i^c \mid E^c)$

a) $P(E \mid I_n)$, $I_n = T_1 \cap T_2 \cap \ldots \cap T_5$ (1)

$$P(E \mid I_n) = \frac{P(E \cap I_n)}{P(I_n)} = \frac{P(I_n \mid E) \cdot P(E)}{P(I_n \cap E) + P(I_n \cap E^c)} \text{ (2)} =$$

$$= \frac{P(I_n \mid E) \cdot P(E)}{P(I_n \mid E) \cdot P(E) + P(I_n \mid E^c) \cdot P(E^c)} = \frac{P(T_1 \mid E) \cdot \ldots \cdot P(T_n \mid E) \cdot P(E)}{P(T_1 \mid E) \ldots P(T_n \mid E) \cdot P(E) \text{ (4)} + P(T_1 \mid E^c) \ldots P(T_n \mid E^c) \cdot P(E^c) \text{ (4)}}$$

b) updating the formula sequentially

$$P(E \mid I_{k+1}) = \frac{P(E \cap I_{k+1})}{P(I_{k+1})}$$

Recall: $I_{k+1} = T_1 \cap \ldots \cap T_{k+1}$

$P(I_{k+1} \mid E) = P(T_1 \cap \ldots \cap T_{k+1} \mid E) = P(T_1 \cap \ldots \cap T_k \mid E) \cdot P(T_{k+1} \mid E) = P(I_k \mid E) \cdot P(T_{k+1} \mid E)$

conditional independence

numerator:

$$(*) \quad P(E \cap I_{k+1}) = P(I_{k+1} \mid E) \cdot P(E) = \underbrace{P(I_k \mid E) \cdot P(E)}_{P(I_k \cap E) \text{ (1)}} \cdot P(T_{k+1} \mid E) =$$

$$= P(I_k \cap E) \cdot P(T_{k+1} \mid E) = \underbrace{P(E \mid I_k) \cdot P(I_k)}_{\text{(1)}} \cdot P(T_{k+1} \mid E)$$

denominator:

$$P(I_{k+1}) = P(I_{k+1} \cap E) + P(I_{k+1} \cap E^c) \text{ (2)} = \cancel{P(I_{k+1} \cap E)} \; (*) + (**)$$

$$(**) \quad P(I_{k+1} \cap E^c) = P(I_{k+1} \mid E^c) \cdot P(E^c) = \underbrace{P(I_k \mid E^c) \cdot P(E^c)}_{\text{(1)}} \cdot P(T_{k+1} \mid E^c) =$$

condit. independence

$$= P(I_k \cap E^c) \cdot P(T_{k+1} \mid E^c) = \underbrace{P(E^c \mid I_k) \cdot P(I_k)}_{\text{(1)}} \cdot P(T_{k+1} \mid E^c)$$

$$P(E \mid I_{k+1}) = \frac{(*)}{(*) + (**)} = \frac{P(E \mid I_k) \cdot P(T_{k+1} \mid E) \cdot \cancel{P(I_k)}}{P(E \mid I_k) \cdot P(T_{k+1} \mid E) \cdot \cancel{P(I_k)} + P(E^c \mid I_k) \cdot P(T_{k+1} \mid E^c) \cdot \cancel{P(I_k)}} =$$

$$= \frac{P(E \mid I_k) \cdot P(T_{k+1} \mid E)}{P(E \mid I_k) \cdot P(T_{k+1} \mid E) + P(E^c \mid I_k) \cdot P(T_{k+1} \mid E^c)} \quad (***)$$

c) Do a couple by hand then using code (see slides Tutorial 2)

1) $P(E) = 0.005 \Rightarrow P(E^c) = 1 - 0.005 = 0.995$

$P(T_i \mid E) = 0.99 \Rightarrow P(T_i^c \mid E) = 1 - 0.99 = 0.01$

$P(T_i^c \mid E^c) = 0.99 \Rightarrow P(T_i \mid E^c) = 1 - 0.99 = 0.01$

$T_1 = -$, $T_2 = +$, $T_3 = +$, $T_4 = +$, $T_5 = +$

want $P(E \mid I_k)$, $k = 1, \dots, 5$

$$P(E \mid T_1) = \frac{P(E \cap T_1)}{P(T_1)} = \frac{P(T_1 \mid E) \cdot P(E)}{P(T_1 \cap E) + P(T_1 \cap E^c)} =$$

(1) (2)

$$= \frac{P(T_1 \mid E) \cdot P(E)}{P(T_1 \mid E) \cdot P(E) + P(T_1 \mid E^c) \cdot P(E^c)} = \cancel{\frac{0.99 \cdot 0.005}{0.99 \cdot 0.005 + 0.01 \cdot 0.995}}$$

(1) (1)

But we have $T_1 = -$, so T_1^c

1) $$P(E \mid T_1^c) = P(E \mid I_1) = \frac{P(T_1^c \mid E) \cdot P(E)}{P(T_1^c \mid E) \cdot P(E) + P(T_1^c \mid E^c) \cdot P(E^c)} = \frac{0.01 \times 0.005}{0.01 \times 0.005 + 0.99 \times 0.995} = 5.1 \times 10^{-5}$$

note that $I_1 = T_1^c$

2) Now $T_2 = +$

$$P(E \mid I_2) = \frac{P(E \mid T_1^c) \cdot P(T_2 \mid E)}{P(E \mid T_1^c) \cdot P(T_2 \mid E) + P(E^c \mid T_1^c) \cdot P(T_2 \mid E^c)} \quad (***)$$

$$P(E \mid I_2) = \frac{5.1 \times 10^{-5} \cdot 0.99}{5.1 \times 10^{-5} \cdot 0.99 + (1 - 5.1 \times 10^{-5}) \cdot 0.01} = 0.005$$

note $I_2 = T_1^c \cap T_2$

3) $T_3 = +$

Again, using (***)

$$P(E \mid I_3) = \frac{0.005 \cdot 0.99}{0.005 \cdot 0.99 + (1 - 0.005) \cdot 0.01} = 0.332$$

$I_3 = T_1^c \cap T_2 \cap T_3$

4) $T_4 = +$ same way as before

$$P(E \mid T_4) = \frac{0.332 \cdot 0.99}{0.332 \cdot 0.99 + (1 - 0.332) \cdot 0.01} = 0.9801$$

$I_4 = T_1^c \cap T_2 \cap T_3 \cap T_4$

5) $T_5 = +$ same way as before:

$$P(E \mid T_5) = \frac{0.9801 \cdot 0.99}{0.9801 \cdot 0.99 + (1 - 0.9801) \cdot 0.01} = 0.9997$$

We can write code for these calculations (see slides)

+ extension of the problem

c 2) $P(E \mid I_1) = 0.3322$

$P(E \mid I_2) = 0.9801$

$P(E \mid I_3) = 0.9997$

$P(E \mid I_4) = 0.9999$

$P(E \mid I_5) = 0.9997$ the same as in c(1)

Random permutations

Definition Permutation is essentially changing order of observations

formally a permutation of $\{1, 2, \ldots, n\}$ is 1-1 f-n g from a set onto itself

Ex 1: $\{1, 2, 3, 4, 5\} \to \{1, 5, 3, 2, 4\}$

Each particular permutation of a set of size n has probability $\frac{1}{n!}$ ($n!$ total possible permutations)

A fix point is a point that stays in place after permut.
in ex 1 1 is a fix point

define $A_i = \{i \text{ is a fix point}\}$

$$P(A_i) = \frac{(n-1)!}{n!} = \frac{1}{n}, \quad i = 1, \ldots, n \quad ; \quad P(A_i \cap A_j) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$$

Question What is $\mathbb{P}$ there are no fix points

$A_i = \{i \text{ is a fix point}\}$
$B_n^0 = \{\text{there are no fix points}\}$

inclusion-exclusion

$$B_n^0 = \left(\bigcup_{i=1}^{n} A_i\right)^c$$

$$(*)\ \mathbb{P}(B_n^0) = 1 - \mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = 1 - \sum_{i=1}^{n} \mathbb{P}(A_i) + \sum_{i,j} \mathbb{P}(A_i \cap A_j) - \sum_{i,j,k} \mathbb{P}(A_i \cap A_j \cap A_k) + \ldots =$$

$$= 1 - \sum_{j=1}^{n} (-1)^{j-1} \sum_{|J_n| = j} P\left(\bigcap_{i \in J_n} A_i\right)$$

this means a set of indicies of size j

for $j = 2$ we get $\sum_{i,j} \mathbb{P}(A_i \cap A_j)$

for $j = 3$ get $\sum_{i,j,k} \mathbb{P}(A_i \cap A_j \cap A_k)$

and so on

$$\sum_{|J_n| = j} \mathbb{P}\left(\bigcap_{i \in J_n} A_i\right) = \binom{n}{j} \cdot \frac{(n-j)!}{n!} = \frac{n!}{j!\,(n-j)!} \cdot \frac{(n-j)!}{n!} = \frac{1}{j!}$$

probability of each combination

for each j there are as many combinations

$$\therefore\ (*) = 1 - \sum_{j=1}^{n} (-1)^{j-1} \cdot \frac{1}{j!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots \approx e^{-1}$$

if $n \to \infty$

Want: Probability of $1 \le k \le n$ fix points

For any k the probability that they are the only fix points

$$\underbrace{\frac{1}{n(n-1)\cdots(n-k+1)}}_{\mathbb{P}(k \text{ pts are fixed})} \quad \underbrace{\mathbb{P}(B^0_{n-k})}_{0 \text{ fixed points in the rest } n-k}$$

Since there are $\binom{n}{k}$ possible combinations of k points

$$\mathbb{P}(k \text{ fix points}) = \binom{n}{k} \frac{1}{n(n-1)\cdots(n-k+1)} \mathbb{P}(B^0_{n-k}) = \frac{\cancel{n!}}{k! \underbrace{(n-k)! \; n(n-1)\cdots(n-k+1)}_{\cancel{n!}}} \mathbb{P}(B^0_{n-k})$$

$$= \frac{1}{k!} \sum_{j=0}^{n-k} (-1)^{j-1} \frac{1}{j!}$$

Interesting information: We cannot have $n-1$ fixed points, only n or $n-2$ because once $(n-1)$ points are fixed all n points have to be fixed