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Tutorial 5

October 13th, 201

Gaussian / Normal distribution

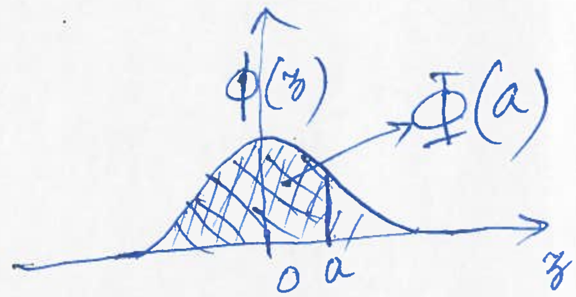
$X \sim N(\mu, \sigma^2)$

X → r.v. ; μ → mean ; σ^2 → variance

pdf $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $-\infty < x < \infty$

$$\boxed{Z = \frac{X-\mu}{\sigma}}$$

σ → standard dev.



$Z \sim N(0,1)$

Z → std. normal random variable.

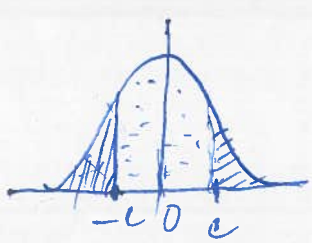
$N(0,1)$ → standard normal distribution

pdf $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, $-\infty < z < \infty$

cdf $\Phi(z) = P(Z \le z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\, dt$ *

properties :-

a) all properties that any CDF satisfies.



~~1) $\Phi(-c) = 1 - \Phi(c)$~~

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2) $\Phi(0) = 0.5$

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Examples or application of normal distribution.

a) estimation error modeling

b) Additive noise (communication base-band modeling)

* Thermal noise, interference

c) Multipath fading (complex Gaussian)

Central Limit Theorem

If $X_1, X_2, \ldots$ are iid (independent identically distributed) random variables, each having mean μ and variance σ^2, then define:

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i ,$$

$$S_n = \sum_{i=1}^{n} X_i = n\bar{X}_n ;$$

standardization

$$\boxed{Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}}$$

Equivalent

a) $S_n \xrightarrow{D} N(n\mu, n\sigma^2)$

b) $\bar{X}_n \xrightarrow{D} N(\mu, \frac{\sigma^2}{n})$

c) $Z_n \xrightarrow{D} N(0,1)$

X_i's are independent

$$\bar{X}_n = \frac{1}{n}[X_1 + X_2 + \cdots + X_n]$$

$$var(\bar{X}_n) = \frac{1}{n^2} var(X_1) + \frac{1}{n^2} var(X_2) + \cdots + \frac{1}{n^2} Var(X_n)$$

$$= \frac{1}{n^2}\cdot(n\sigma^2) = \frac{\sigma^2}{n}.$$

Example: Insurance claims

Suppose that an insurance company has 10,000 policy holders. The expected yearly claim per policyholder is \$240 with a standard deviation of \$800. What is the approximate probability that the total yearly claims $S_{10,000} >$ \$2.6 Million

$$E[S_{10,000}] = 10,000 \times 240 = 2,400,000$$
$$\mathrm{SD}(S_{10,000}) = \sqrt{10,000} \times 800 = 80,000$$

$$\begin{aligned} &P[S_{10,000} > 2,600,000] \\ &\quad = P\left[\frac{S_{10,000} - 2,400,000}{80,000} > \frac{2,600,000 - 2,400,000}{80,000}\right] \\ &\quad \approx P[Z > 2.5] = 0.0062 \end{aligned}$$

Note that this probability statement does not use anything about the distribution of the original policy claims except their mean and standard deviation. Its probable that their distribution is highly skewed right (since $\mu_x << \sigma_x$), but the calculations ignore this fact.

```
% Central Limit Theorem demo
clear all
close all
clc
n = 100; % Number of iid r.v.'s
for loop = 1:n
   X(loop,:) = rand(1,1e5); % uniform distribution
end
X_n_bar = mean(X,1);
Z_n = (X_n_bar-0.5)/(sqrt(1/12/n)); % Standardization
[val1,idx1] = hist(X_n_bar,50);
[val2,idx2] = hist(Z_n,50);
figure;bar(idx1,val1/sum(val1)) % “Un-standardized mean” pdf
figure;bar(idx2,val2/sum(val2)) % “Standardized mean” pdf ~ Std normal distribution
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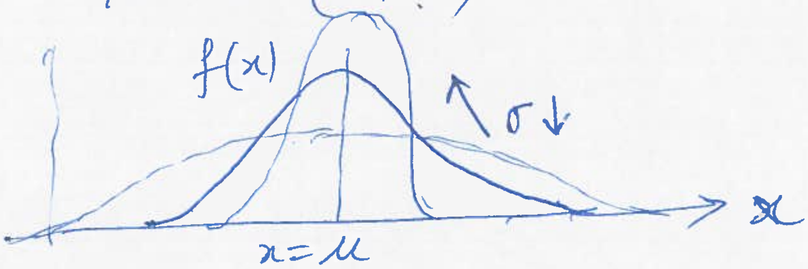
$X \sim N(\mu, \sigma^2)$



Problem 1. (Module 4, pp. 24)

$$Z \sim N(0,1)$$

a) $P(0.1 \le Z \le 0.35)$

$= \Phi(0.35) - \Phi(0.1)$

$= 0.097$

b) $P(Z > 1.25)$

$= 1 - P(Z \le 1.25)$

$= 1 - \Phi(1.25)$

$= 0.1056$

* $P(Z \ge c)$

$= P(Z > c)$

$[\because P(Z=c) = 0]$

(c), (d) solve yourselves.

(e) $P(|Z| < c) = 0.95$

$\Rightarrow P(-c \le Z < c) = 0.95$

$\Rightarrow \Phi(c) - \Phi(-c) = 0.95$

$\Rightarrow \Phi(c) - \{1 - \Phi(c)\} = 2\Phi(c) - 1 = 0.95$

(4) $2\Phi(c) - 1 = 0.95$

$\Rightarrow \Phi(c) = \frac{1.95}{2}$

$\Rightarrow c = \Phi^{-1}\left(\frac{1.95}{2}\right)$

$= 1.96$.

Problem 2

$X \sim N(\overset{\mu}{3}, \overset{\sigma^2}{25})$

$\sigma = \sqrt{\text{variance}}$

a) $P(X > 4)$

$= P\left(\frac{X - \overset{3}{\mu}}{\underset{=5}{\sigma}} > \frac{4-3}{5}\right)$

$= P\left(Z > \frac{1}{5}\right)$

$= 1 - P\left(Z \le \frac{1}{5}\right)$

$= 1 - \Phi(0.2)$

$= 0.4207$.

(b) $P(2 < X < 4)$

$= P\left(\frac{2-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{4-\mu}{\sigma}\right)$

$= P\left(\frac{2-3}{5} < Z < \frac{4-3}{5}\right)$

$= P(-0.2 < Z < 0.2)$

(5) $P(-0.2 < Z < 0.2)$ $\left[* \; P(a \le Z \le b) = \underline{\Phi(b) - \Phi(a)} \right]$

$= \Phi(0.2) - \Phi(-0.2)$

$= \Phi(0.2) - \{1 - \Phi(0.2)\}$

$= 2\Phi(0.2) - 1$

$= 0.1585.$

Solve (c), (d) & (f) yourselves.

(e) $P(|X - \overset{\mu}{3}| < c) = 0.95$

$\Rightarrow P\left(\left|\frac{X-3}{\sigma}\right| < \frac{c}{\sigma}\right) = 0.95$ $\boxed{\sigma = 5}$

σ is positive

$\Rightarrow P\left(|Z| < \frac{c}{\sigma}\right) = 0.95$

$\Rightarrow P\left(-\frac{c}{\sigma} < Z < \frac{c}{\sigma}\right) = 0.95$

$\Rightarrow 2\Phi\left(\frac{c}{\sigma}\right) - 1 = 0.95$

$\Rightarrow \frac{c}{5} = \Phi^{-1}\left(\frac{1.95}{2}\right)$

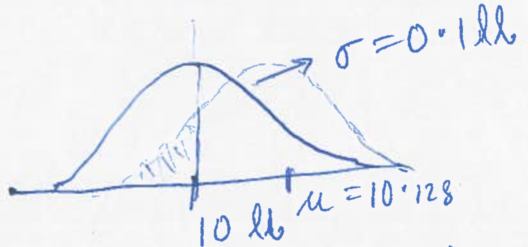
$= 1.96$

$\Rightarrow c = 5 \times 1.96 = 9.8$

(6)

Problem 3 :

(a)

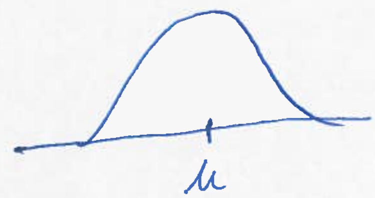


$X \sim N(\mu, (0{\cdot}1)^2)$

X → actual weight.

** eg. If the machine operator sets the mean = 10.

X :→



$$P(X < 10) \le 10\% = 0{\cdot}1$$

$$\Rightarrow P\left(\frac{X-\mu}{\sigma} < \frac{10-\mu}{\sigma}\right) \le 0{\cdot}1$$

$$\Rightarrow P\left(Z < \frac{10-\mu}{0{\cdot}1}\right) \le 0{\cdot}1$$

[eg. ** If $\mu = 10$, $P(X < 10) = ?$
$P(X \le 10) = P(X \ge 10) = 0{\cdot}5$]

$$\Rightarrow \Phi\left(\frac{10-\mu}{0{\cdot}1}\right) \le 0{\cdot}1$$

$$\Rightarrow \frac{10-\mu}{0{\cdot}1} \le \Phi^{-1}(0{\cdot}1)$$

[$\because \Phi(z)$ is monontonically increasing]

$$\Rightarrow \mu \ge 10{\cdot}128$$

7 (b) Solve it yourself.

Review from midterm slides:

Problem: Ball bearing problem.

X : r.v. that represents actual diameter.

μ : intended diameter.

$$P(|X-\mu| < 0.5) \geq 0.9$$

$$\Rightarrow P\left(\left|\frac{X-\mu}{\sigma}\right| < \frac{0.5}{\sigma}\right) \geq 0.9$$

$\sigma > 0$

$$\Rightarrow P\left(|Z| < \frac{0.5}{\sigma}\right) \geq 0.9$$

$$\Rightarrow P\left(-\frac{0.5}{\sigma} < Z < \frac{0.5}{\sigma}\right) \geq 0.9$$

$$\Rightarrow \Phi\left(\frac{0.5}{\sigma}\right) - \Phi\left(-\frac{0.5}{\sigma}\right) \geq 0.9$$

$$\Rightarrow 2\Phi\left(\frac{0.5}{\sigma}\right) - 1 \geq 0.9 = 2\Phi(1.645) - 1$$

$$\Rightarrow \Phi\left(\frac{0.5}{\sigma}\right) \geq \Phi(1.645)$$

$$\Rightarrow \frac{0.5}{\sigma} \geq 1.645$$

$$\Rightarrow \sigma \leq \frac{0.5}{1.645} = 0.304$$

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Tutorial Problem Set B

B.4

$$X : \begin{cases} f_X(x) \\ F_X(x) \end{cases}$$

$$Y = a + bX$$

$$F_Y(y) = P(Y \le y)$$
$$= P(a + bX \le y)$$
$$= \begin{cases} P\left(X \le \frac{y-a}{b}\right), & b > 0 \\ P\left(X \ge \frac{y-a}{b}\right), & b < 0 \end{cases}$$

$$\Rightarrow F_Y(y) = \begin{cases} F_X\left(x = \frac{y-a}{b}\right), & b > 0 \\ 1 - F_X\left(x = \frac{y-a}{b}\right), & b < 0 \end{cases}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$
$$= \frac{d}{dx} F_Y(y) \cdot \frac{d}{dy}\left(x = \frac{y-a}{b}\right)$$

$\left[\frac{d}{dx} F_X(x) = f_X(x)\right]$

$\left[\frac{d}{dy} F_Y(y) = f_Y(y)\right]$

$$= \begin{cases} \cancel{f_X} \left(x = \frac{y-a}{b}\right) \cdot \frac{1}{b}, & b > 0 \\ -f_Y\left(x = \frac{y-a}{b}\right) \cdot \frac{1}{b}, & b < 0 \end{cases}$$

$$= \frac{1}{|b|} f_Y\left(\frac{y-a}{b}\right).$$

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Problem B.5:

Z : standard normal r.v.

(a) $P(-1 < Z < 1)$

$= 2\Phi(1) - 1 = 2 \times 0.8413 - 1$

Compute the other ones yourselves.

(b) $P(|Z| < c_1) = 0.8$

$\Phi(c_2) = P(Z < c_2) = 0.8$

$\Phi(c_2) = 0.8 = P(|Z| < c_1)$

$= P(-c_1 < Z < c_1)$

$= \Phi(c_1) - \Phi(-c_1)$

$\leq \Phi(c_1)$

i.e., $\Phi(c_2) \leq \Phi(c_1)$

$\Rightarrow c_2 \leq c_1$ [$\because \Phi$ is monotonically increasing]

10)

Problem B.6

$X \sim N(\mu, \sigma^2)$

(a) $P(|X-\mu| < 0.01) \geq 0.95$

$\Rightarrow P\left(\left|\frac{X-\mu}{\sigma}\right| < \frac{0.01}{\sigma}\right) \geq 0.95$

$\Rightarrow P\left(|Z| < \frac{0.01}{\sigma}\right) \geq 0.95 = P(|Z|<1.96)$

$\Rightarrow 2\Phi\left(\frac{0.01}{\sigma}\right) - 1 \geq 0.95 = 2\Phi(1.96) - 1$

$\Rightarrow \frac{0.01}{\sigma} \geq 1.96$

$\Rightarrow \sigma \leq \frac{0.01}{1.96}.$

(b) $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

$\boxed{\sigma = 2}$

$X_i \sim N(\mu, \sigma^2)$ $\to \sigma_0^2$

$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

$\Rightarrow P(|\bar{X}-\mu| < 0.01) \geq 0.95$

$\Rightarrow P\left(\left|\frac{\bar{X}-\mu}{\sigma_0}\right| < \frac{0.01}{\sigma_0}\right) \geq 0.95$

$\Rightarrow P\left(|Z| \leq \frac{0.01}{\sigma/\sqrt{n}}\right) \geq 0.95$

$\Rightarrow 2\Phi\left(\frac{\sqrt{n} \cdot 0.01}{2}\right) - 1 \geq 0.95$
$= 2\Phi(1.96) - 1$

$\Rightarrow \sqrt{n} \cdot \frac{0.01}{2} \geq 1.96$

Solve n!!