ELEC 321 / STAT 321

Tutorial Part A

Problem A.1: Let $P(A) = P(B) = 1/3$ and $P(A \cap B) = 1/10$. Find the following (a) $P(B^c)$ (b) $P(A \cup B^c)$ (c) $P(B \cap A^c)$ (d) $P(A^c \cup B)$.

Problem A.2: A track star will run two races. The probability that he wins the first race is 0.70, the probability that he wins the second race is 0.60 and the probability that he wins both races is 0.50. Find the probability that (a) he wins at least one race; (b) he wins exactly one race; (c) He wins neither race; (d) he wins the second race given he lost the first.

Problem A.3: Let $P(A) = 1/2$, $P(B) = 1/8$ and $P(C) = 1/4$, where A, B and C are mutually exclusive. Find (a) $P(A \cup B \cup C)$ (b) $P(A^c \cup B^c)$ (c) $P(A^c \cap B^c)$.

Problem A.4: A box contains three "good" cards and two "bad" (penalty) cards. Player A chooses one card and then player B chooses a card. Compute
(a) P(A good) (b) P(B good | A good) (c) P(B good | A bad) (d) P(A good $\cap$ B good) (e) P(B good) (f) P(A good | B good).

Problem A.5: A box contains five green balls, three black balls and seven red balls. Two balls are selected at random without replacement from the box. What is the probability that
(a) Both balls are red? (b) Both balls are the same color? (c) The second ball is green?

Problem A.6: A family has two children. It is known that at least one of the children is a boy. What is the probability that they have two boys? Assume that the probability of all the possible combinations (boy,boy), (boy,girl), (girl,boy) and (girl,girl) are equally likely.

Problem A.7: Suppose there are n people in a room. What is the probability that at least 2 of them have the same birthday? For what values of n is this probability larger than 1/2? For what values of n is this probability larger than 0.99?

Problem A.8: The random variable X has probability mass function $f(i) = \alpha i$, for $i = 1, 2, ..., 10$. (a) Determine α (b) Calculate $E(X)$ (c) Calculate $P(X \leq 2 | X \leq 5)$.

Problem A.9: John has 10 keys in a chain, one of which opens his apartment door. After a big celebration, he returns home one evening and finds that he cannot identify the apartment key. He works out a clever plan: chooses a key at random and try it. If it fails, he puts it aside and try another randomly chosen key, and continues this way until he can open the door. (a) What is the probability that the first attempt works? (b) What is the probability that the second attempt works? (c) What is the probability that the i^{th} attempt works (for i=3,4,...,10) (d) What is the expected number of attempts until a key works?

Problem A.10: Peter is in the same situation. He comes up with a less clever plan: randomly chooses one key from the chain until the key works. He misses the clever step of setting aside failing keys! (a) What is the probability that the first attempt works? (b) What is the probability that the attempt works? (c) What is the probability that the i^{th} attempt works (for i=3,4,...)? (d) What is the expected number of attempts until the key works?

Problem A.11: A patient goes to a clinic to check whether she has a nasty disease. The doctor orders a battery of 5 tests. Let's introduce the events

$$E = \{\text{ patient sick }\}$$

and

$$T_i = \{\text{ result test } i\}\,, \quad i = 1, 2, ..., n$$

Suppose that the events T_i ($i = 1, 2, ..., n$) are conditionally independent given E and also given E^c. Suppose that $P(E)$ (incidence of the disease), $P(T_i = +|E)$ (sensitivity of the ith test) and $P(T_i = -|E^c)$ (specificity of the ith test) are given.

(a) Derive the formula for $P(E \mid I_n)$ where $I_n = T_1 \cap T_2 \cap \cdots \cap T_n$.

(b) Updating information sequentially. Show that

$$P(E \mid I_{k+1}) = \frac{P(T_{k+1}|E)\,P(E|I_k)}{P(T_{k+1}|E)\,P(E|I_k) + P(T_{k+1}|E^c)\,P(E^c|I_k)}, \quad i = 1, 2, ..., 4$$

(c) Suppose that $P(E) = 0.005$ (incidence of the disease), $P(T_i = +|E) = 0.99$ (sensitivity of the ith test), $P(T_i = -|E^c) = 0.99$ (specificity of the ith test).

(c1) Suppose that $T_1 = -$, $T_2 = +$, $T_3 = +$, $T_4 = +$ and $T_5 = +$. Calculate $P(E \mid I_k)$ for k=1, 2, 3, 4 and 5.

(c2) Suppose that $T_1 = +$, $T_2 = +$, $T_3 = +$, $T_4 = +$ and $T_5 = -$. Calculate $P(E \mid I_k)$ for k=1, 2, 3, 4 and 5.

(c3) Suppose now that $T_1 = +$, $T_2 = -$, $T_3 = -$, $T_4 = -$ and $T_5 = -$. Calculate $P(E \mid I_k)$ for k=1, 2, 3, 4 and 5.

(c4) Suppose now that $T_1 = -$, $T_2 = -$, $T_3 = +$, $T_4 = -$ and $T_5 = +$. Calculate $P(E \mid I_k)$ for k=1, 2, 3, 4 and 5.

Comparing the results for (c1) and (c2) and the results for (c3) and (c4) suggests that value of $P(E \mid I_5)$ only depends on the number of positive tests but not on their order. Is that true in general for this setting? Why?

Problem A.12: For a majority decoding algorithm, if majority of the (2N+1) transmitted identical digits are received correctly, then the received digit is considered correctly decoded. Let X be the number of errors in the transmission of the (2N+1) transmitted identical digits, and p as the probability that each of the (2N+1) bits can be decoded correctly on its own. Assume that the errors in each of the (2N+1) positions are independent of each other. (a) If N=2, p=0.2, what is the probability that one transmitted bit using majority decoding algorithm is decoded correctly? (b) If we only want use 3 identical bits in the majority decoding algorithm, what is the minimum p required to have a better performance compared to (a)? (c) If we use 7 identical bits, repeat (b).

Problem A.13: A five card poker hand is dealt from a shuffled deck of 52 cards. (a) A full house has the pattern AAABB, what is the probability of a full house? (b) What is the probability that the hand contains two pairs (i.e., two cards of one rank, two of another rank and a fifth card of yet another rank)?

Problem A.14: Let X be a binomial random variable with parameters n and p. What value of p maximizes P(X=k) for a given value of k in in 0, 1, . . . , n?

Problem A.15: Let X be a Poisson random variable with parameter λ . Find the value of λ which maximizes P(X = k) for a given non-negative integer k.

Problem A.16: During the test of bit-error-rate(BER) for communication systems, an erroneous bit occurs at a rate $\lambda = 3$ per hour. (a) What is the probability of receiving 5 erroneous bits in the next two hours? (b) What is the probability of not detecting any erroneous bit in a 30-minute periods? (c) What is the expected waiting time until the first erroneous bit is detected? The variance? (d) What is the expected number of erroneous bits in the next 10 minutes?

Problem A.17: When transmitting data through a noisy channel, a binary (0 or 1) message transmitted through a noisy communication channel is received incorrectly with probability r_0 and r_1, respectively. Errors in different symbol transmissions are independent. The channel source transmits a 0 with probability p and transmits a 1 with probability 1-p. (a)What is the probability that a randomly chosen symbol is received correctly? (b) Suppose that the string of symbols 1011 is transmitted. What is the probability that all the symbols in the string are received correctly? (c) In an effort to improve reliability, each symbol is transmitted three times and the received symbol is decoded by majority rule. In other words, a 0 (or 10 is transmitted as 000 (or 111, respectively), and it is decoded at the receiver as a 0 (or 1) if and only if the received three-symbol string contains at least two 0s (or 1s, respectively). What is the probability that a transmitted 0 is correctly decoded? (d) For what values of r_0 is this an improvement over sending a single 0? (e) Suppose that the scheme of part (c) is used. What is the probability that a 0 was transmitted given that the received string is 101?