1

Tutorial 2 (Sep 22, 2017)

1, 4, 6, 7

Problem Set A : 2, 5, 8, 9, ~~10~~

Lecture Slides (Module 2) pp. 53 - end.

Basic R code (3 prisoners problem)

Problem A.2 :

W_1 : event that he wins the first race

W_2 : " " " " " 2nd "

$P(W_1) = 0.7$

$P(W_2) = 0.6$, $P(W_1 \cap W_2) = 0.5 \rightarrow \neq P(W_1) \cdot P(W_2)$

Venn diagram

a) $P(W_1 \cup W_2)$

Event that at least one race is won.

$E_1 = W_1 \cup W_2 = (W_1 \cap W_2^c) \cup (W_1^c \cap W_2) \cup (W_1 \cap W_2)$

$P(W_1 \cup W_2) = P(W_1) + P(W_2) - P(W_1 \cap W_2)$

$= 0.7 + 0.6 - 0.5 = 0.8$

(b) 2

Winning exactly one race

$$E_2 = (W_1 \cap W_2^c) \cup (W_1^c \cap W_2)$$

$$P(W_1 \cap W_2^c) = P(W_1) - P(W_1 \cap W_2)$$
$$= 0.7 - 0.5$$
$$= 0.2$$

$$P(W_1^c \cap W_2) = P(W_2) - P(W_1 \cap W_2)$$
$$= 0.6 - 0.5 = 0.1$$

$$P(E_2) = 0.2 + 0.1 = 0.3$$

<u>Alterative way:</u>

$$P(E_2) = P(E_1) - P(W_1 \cap W_2)$$
$$= 0.8 - 0.5$$
$$= 0.3$$

(c) E_3 = he wins neither race

$$= W_1^c \cap W_2^c = (W_1 \cup W_2)^c$$

$$\boxed{\begin{aligned}(A \cup B)^c &= A^c \cap B^c \\ (A \cap B)^c &= A^c \cup B^c\end{aligned}}$$

$$P(E_3) = 1 - P(\underbrace{W_1 \cup W_2}_{E_1}) \rightarrow \text{part (a)}$$
$$= 0.2$$

(d) E_4 = He wins 2nd race, given that he lost the first

$$P(E_4) = P(W_2 \mid W_1^c) = \frac{P(W_2 \cap W_1^c)}{P(W_1^c)} = \frac{0.1}{0.3} = \cdot$$

2

Problem A.5

For $i = 1, 2$

15 balls : 5 Green, 3 black, 7 Red

R_i : event that i^{th} draw is a red ball

B_i = i^{th} draw is bla

G_i = i^{th} draw is gree

(a) $P(R_1 \cap R_2)$

Without replacement.

$$= P(R_1) \cdot P(R_2 | R_1)$$

$$= \frac{\cancel{7}}{\cancel{15}\,5} \times \frac{\cancel{6}\,3}{\cancel{14}\,2} = \frac{1}{5}.$$

(b) $P(R_1 \cap R_2) + P(B_1 \cap B_2) + P(G_1 \cap G_2)$

$$= \frac{1}{5} + P(B_1) \cdot P(B_2 | B_1) + P(G_1) \cdot P(G_2 | G_1)$$

$$= \frac{1}{5} + \frac{\cancel{3}}{\cancel{15}\,5} \times \frac{2}{\cancel{14}\,7} + \frac{\cancel{5}}{\cancel{15}\,3} \times \frac{\cancel{4}\,2}{\cancel{14}\,7}$$

$$= \frac{1}{5} + \frac{1}{35} + \frac{2}{21}$$

$$= \frac{21 + 3 + 10}{105} = \frac{34}{105}$$

(c) $P(G_2) = P(G_2 \cap R_1) + P(G_2 \cap B_1) + P(G_2 \cap G_1)$

$\hookrightarrow$ Theorem of total probability

$$= P(R_1) \cdot P(G_2 | R_1) + P(B_1) \cdot P(G_2 | B_1) + \frac{2}{21}$$

$$= \frac{7}{15} \cdot \frac{5}{14} + \frac{3}{15} \times \frac{5}{14} + \frac{2}{21} \rightarrow \text{Simplify}$$

4) $P(G_2) = P(G_2 \cap G_1) + P(G_2 \cap G_1^c)$

Alterative way !

$$= \frac{2}{21} + P(G_1^c) \cdot P(G_2 | G_1^c)$$

$$= \frac{2}{21} + \frac{\cancel{10}^{5}}{\cancel{15}_{3}} \times \frac{\cancel{5}}{\cancel{14}_{7}}$$

$$= \frac{2}{21} + \frac{5}{21} = \frac{1}{3}.$$

Problem A.8

X is a random variable.

pmf $\rightarrow f(i) = P(X = i) = \alpha i,$

$i = 1(1)10.$

Properties of $f(i)$
$\begin{cases} a) & f(i) \geq 0. \\ b) & \sum_i f(i) = 1 \end{cases}$

(a) $\sum_{i=1}^{10} f(i) = \alpha \sum_{i=1}^{10} i = \alpha . \frac{10 \times 11}{2} = 55\alpha.$

Sum of first n positive integers.

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

1) Prove by induction.

or, 2)

$$S_n = 1+2+3+\cdots+n$$

$$S_n = n+(n-1)+(n-2)+\cdots+1$$

$$2S_n = \underbrace{(n+1)+(n+1)+\cdots+(n+1)}_{n \text{ times}} \Rightarrow S_n = \frac{n(n+1)}{2}$$

~~$S_n = \frac{n(n+1)}{2}$~~

$$\sum_{i=1}^{10} f(i) = 55\alpha = 1$$

$$\Rightarrow \boxed{\alpha = \frac{1}{55}}$$

(b) $E(X) = \sum_{i} x_i \, P(X = x_i)$ —— (i)

In this case,

$$E(X) = \sum_{i=1}^{10} i \cdot (\alpha i)$$

$$= \frac{1}{55} \sum_{i=1}^{10} i^2 = \frac{1}{55} \cdot \frac{10 \times 11 \times 21}{6} = \underline{\underline{7}}$$

~~Sum of~~ $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$

a) Induction

b) Using $(n+1)^3$ expansion.

Eq. (i) :

For a discrete random variable X that has the following pmf (probability mass function)

$$P(X = x_i) = f(x_i)$$

$E(X)$: Expectation of X (weighted avg.) $= \sum_{i} \underbrace{x_i}_{\text{value}} \underbrace{f(x_i)}_{\text{weight}}$

(b)

e.g. X is random variable that takes the values $x_1, x_2, \dots x_n$ with equal probabilities, i.e., $\frac{1}{n}$.

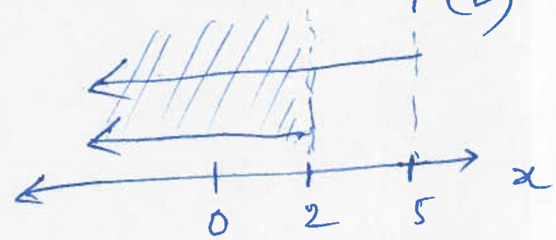
$$E(X) = \sum_{i=1}^{n} x_i \frac{1}{n}$$

$$= \frac{1}{n} \sum x_i \longrightarrow \text{average.}$$

(c)

$$P(X \leq 2 | X \leq 5)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$



$$= \frac{P(X \leq 2 \cap X \leq 5)}{P(X \leq 5)}$$

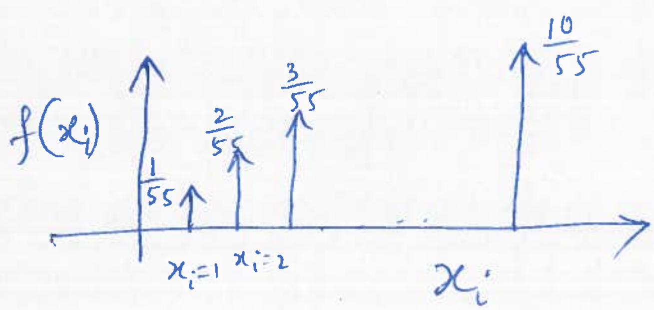
$$= \frac{P(X \leq 2)}{P(X \leq 5)}$$

$$= \frac{P(X=1) + P(X=2)}{\sum_{i=1}^{5} P(X=i)}$$

$$= \frac{\frac{1}{55} \cdot 1 + \frac{1}{55} \cdot 2}{\frac{1}{55} \sum_{i=1}^{5} i} = \frac{3/55}{15/55}$$

$$= \frac{1}{5}.$$

* $f(i) = \alpha i$



Problem A-9

(a) K_i : i^{th} attempt works.
event

$$P(K_1) = \frac{1}{10}$$

(b)
$$P(K_2) = P(\cancelto{0}{K_2 \cap K_1}) + P(K_2 \cap K_1^c)$$
$$= P(K_1^c) \cdot P(K_2 | K_1^c)$$
$$= \frac{9}{10} \times \frac{1}{9} = \frac{1}{10}$$

(c)
$$P(K_i) = P(K_i \cap K_{i-1}^c \cap K_{i-2}^c \cap \cdots \cap K_1^c) \quad \text{(ii)}$$

Chain rule of probability:
$$P\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P\left(A_i \,\Big|\, \bigcap_{j=1}^{i-1} A_j\right)$$

Special case:
A_1 & A_2
$$P(A_1 \cap A_2) = P(A_1) . P(A_2 | A_1)$$

Applying chain rule to (ii) –
$$P(K_i) = P(K_i | K_{i-1}^c K_{i-2}^c \cdots K_1^c) P(K_{i-1}^c | K_{i-2}^c K_{i-3}^c \cdots K$$
$$\cdots P(K_2^c | K_1^c) . P(K_1^c)$$

$$= \frac{1}{\cancel{10-i+1}} \cdot \frac{\cancel{10-i+1}}{10-i+2} \cdots \frac{8}{\cancel{9}} \cdot \frac{\cancel{9}}{10} = \frac{1}{10}$$

(d) $E(X) = \sum_{i=1}^{10} i\, P(K_i)$

No. of attempts

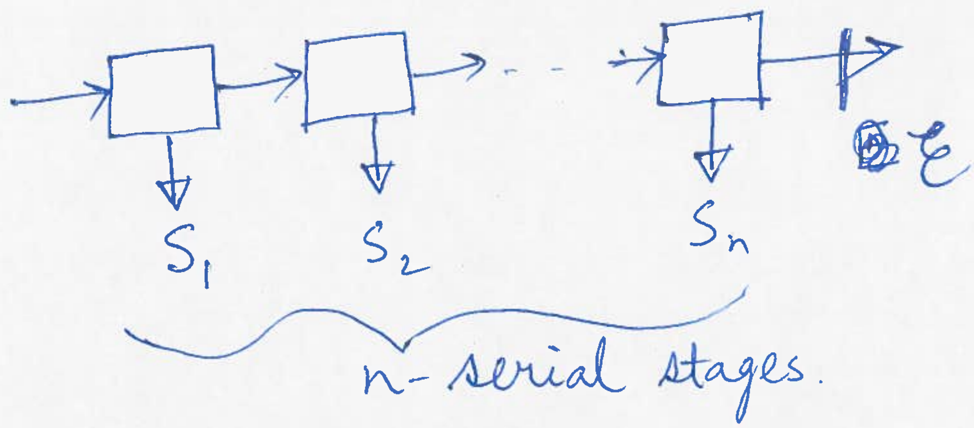
$$= \frac{1}{10}\sum_{i=1}^{10} i = 5.5.$$

<u>Interpretation</u>: If John does this trick for a long time, on an average, he should be able to open the door after 5^{th} or 6^{th} attempt

Lecture Slides (Module 2)

pp. 53 to end.

Recursive Bayesian / Sequential Bayesian method.



$S_i = \begin{Bmatrix} + , - \\ 1 , 0 \end{Bmatrix}, \; i = 1:n.$

e.g. 1 : To determine whether a component of a device is faulty or not, based on n experiments:

e.g. 2   Whether a patient is sick based on n blood-tests.

e.g. 3   Spam e-mail detection.

What is given : a-priori known.

Based on survey on a large no. of samples.

1) $\pi_0 = P(\mathcal{E})$ given

n-stages or tests.

S_i is the binary outcome of the i^{th} test.

2) $p_i = P(S_i = + \mid \mathcal{E}) \rightarrow$ Sensitivity of the i^{th} test

3) $q_i = P(S_i = - \mid \mathcal{E}) \rightarrow$ Specificity of the i^{th} test

4) Outcomes S_i, e.g. $\{+, -, -, + \cdots +\}$

Conditions : Conditional independence of S_i's.

1) $P\left(\bigcap_{i=1}^{n} S_i \mid \mathcal{E}\right) = \prod_{i=1}^{n} P(S_i \mid \mathcal{E})$

a-priori known.

2) $P\left(\bigcap_{i=1}^{n} S_i \mid \mathcal{E}^c\right) = \prod_{i=1}^{2} P(S_i \mid \mathcal{E}^c)$

Objective :

To find $P(\mathcal{E} \mid S_1 \cap S_2 \cap \cdots S_n) = \pi_n$

Solution : Formulate a recursive way to compute $\pi_K = P(\mathcal{E} \mid S_1 \cap S_2 \cdots \cap S$

$K = 1 : n$.

$$\pi_K = \frac{a\,\pi_{K-1}}{a\pi_{K-1} + b(1-\pi_{K-1})}, \quad K = 1(1)n.$$

$$a = \begin{cases} p_k & \text{if } s_k = + \\ 1-p_k & \text{o.w.} \end{cases}$$

$$b = \begin{cases} 1-q_k & \text{if } s_k = + \\ q_k & \text{o.w.} \end{cases}$$

Example of Spam-mail detection:

$$\mathcal{E} = \{ \text{email is spam} \}$$

$$W_i = \{ \text{word } i \text{ is in the message} \}$$

$$S_i = \{ \underset{\underset{1}{+}}{W_i} \text{ or } \underset{\underset{0}{-}}{W_i^c} \}$$

$$\pi_0 = 0.1$$

$$\pi_1 = \frac{p_1 \pi_0}{p_1 \pi_0 + (1-q_1)(1-\pi_0)} = 0.804.$$