ELEC 321 / STAT 321

Tutorial Part B

Problem B.1: A certain event, A, occurs at a rate $\lambda = 3$ per hour. (a) What is the probability of 5 occurrences of A in the next two hours? (b) What is the probability of waiting more than 30 minutes for the next occurrence of A?(c) What is the expected waited time until the first occurrence of A? The variance? (d) What is the expected number of occurrences of A in the next 10 minutes? (e) Ten minutes went by and A didn't occur. What is the probability that A will not occur in the next 5 minutes?

Problem B.2: Suppose that the random variable X has (Pareto) distribution function

$$F(x) = 1 - \frac{1}{x^2}, \qquad \text{for } x > 1$$

and $F(x) = 0$ for $x \leq 1$. (a) Calculate $P(1.5 < X < 2.5)$ (b) Calculate $P(X \leq 3 | X > 2)$ (c) Calculate the mean and standard deviation of X (d) Calculate the median of X, which is denoted by m and defined as the solution to the equation $F(x) = 0.5$. Notice that $P(X < m) = P(X > m) = 1/2$.

Problem B.3: Suppose that the random variable X has (Pareto) distribution function

$$F(x) = 1 - \frac{1}{x^2}, \qquad \text{for } x > 1$$

Find the density function for (a) $Y = X^2$ (b) $V = 1/X$ (c) $W = \ln(X)$. Make sure you first identify the range for each transformation.

Problem B.4: Suppose X has a continuous density function $f_X(x) > 0$, for all x, and distribution function $F_X(x)$. Let

$$Y = a + bX$$

where a and b are given constants. Show that

$$F_Y(y) = F_X\left(\frac{y-a}{b}\right) \qquad \text{and} \qquad f_X(x) = \frac{1}{|b|} f_X\left(\frac{y-a}{b}\right)$$

Problem B.5: Suppose Z is a standard normal random variable. (a) Given that $\Phi(1) = 0.8413$ compute $P(-1 < Z < 1)$, $P(Z > 1)$ and $P(Z < -1)$. (b) Suppose that c_1 is such that $P(|Z| < c) = 0.80$. and c_2 is such that $P(Z < c_1) = 0.80$. Which is larger, c_1 or c_2 ? Why?

Problem B.6: Suppose that $X \sim N(\mu, \sigma^2)$. (a) What is the largest value of σ^2 such that $P(|X - \mu| < 0.01) \geq 0.95$? Now suppose that n independent $N(\mu, 4)$ measurements $X_1, X_2, ..., X_n$ will be averaged to form $\bar{X} = (1/n) \sum_{i=1}^{n} X_i$. What is the smallest n for which $P(|\bar{X} - \mu| < 0.01) \geq 0.95$? You can use that $P(|Z| < 1.96) = 0.95$.

Problem B.7: Given the variables (X, Y) with continuous joint density

$$f(x, y) = 2(x + y), \quad 0 < x < y < 1$$

(a) Find the means μ_X, μ_Y, the variances σ_X^2, σ_Y^2, the covariance σ_{XY} and the correlation coefficient ρ_{XY}. (b) What is the best linear predictor for Y given $X = x$? [**Hint:** the best linear predictor is $L(x) = a + bx$, with $b = Cov(X, Y)/Var(X)$ and $a = \mu_Y - b\mu_Y$].

Problem B.8: Given the variables (X, Y) with continuous joint density

$$f(x, y) = 2(x + y), \quad 0 < x < y < 1$$

What is the best predictor for Y given $X = x$? [**Hint:** use the result from **HW 2:** the best predictor is $\hat{y}(x) = E(Y|X = x)$]

Problem B.9: Suppose that $X_1, X_2, ..., X_n$ are independent continuous random variables with common distribution function $F(x)$ and density $f(x)$. Let

$$U = \max\{X_1, X_2, ..., X_n\}.$$

(a) Find the distribution function, $F_U(u)$, for U. (b) Differentiate $F_U(u)$ to obtain the density, $f_U(u)$. (c) Suppose now that

$$F(x) = 1 - e^{-x}, \qquad \text{for } x > 0$$

and $n = 10$. Calculate $P(1 < U < 2)$.

Problem B.10: Suppose that $X_1, X_2, ..., X_n$ are independent continuous random variables with common distribution function $F(x)$ and density $f(x)$. Let

$$V = \min\{X_1, X_2, ..., X_n\}.$$

(a) Find the distribution function, $F_V(v)$, for V. Comment on your result. (b) What is the density $f_V(v)$? (c) Suppose now that

$$F(x) = 1 - e^{-x}, \qquad \text{for } x > 0$$

and $n = 10$. What are the mean and variance of V? (d) How the mean and the variance evolve as n gets larger?

Problem B.11: [**BPSK Signaling in Digital Communication**] A binary message is transmitted as a signal s, which is either -1 or +1. The communication channel corrupts the transmission with additive normal noise with mean $\mu = 0$ and variance $\sigma^2 = 0.7$. The receiver concludes that the signal is $-1(+1)$ if the value of the recieved signal is < 0 (> 0).

a. What is the probability of error when -1 is sent?

b. What is the probability of error when +1 is sent?

c. If -1 and +1 can be sent with equal probability, what is the combined probability of error?

d. (bonus) Repeat the calculations for general $\sigma^2 > 0$. A widely used upper bound on the tail probability of a standard normal random variable (Z) is as follows:

$$P(Z > z) \leq \frac{e^{-z^2/2}}{z\sqrt{2\pi}}$$

What does this tell you about how the probability of error changes with respect to σ?

Problem B.12: The discrete random variables X and Y have joint pmf

$$f(x, y) = c\left(x^2 + y^2\right), \; x = 1, 2, 4, \; y = 1, 3$$

(a) What is the value of c?
(b) What is $P(X < Y)$?
(c) What is $P(X > Y)$?
(d) What is $P(X = Y)$?
(e) What is $P(Y = 3)$?
(f) Find the marginal pmf's for X and Y.
(g) Find $E(X)$, $E(Y)$ and $E(XY)$
(h) Find $Var(X)$, $Var(Y)$ and $Var(X + Y)$
(i) Find $E(X|X \geq Y)$ and $Var(X|X \geq Y)$

Problem B.13: Suppose that X and Y are independent, identically distributed, geometric random variables with parameter p and range $\{1, 2, ...\}$. Show that for all $n = 2, 3, ...$

$$P(X = i|X + Y = n) = \frac{1}{n-1}, \text{ for } i = 1, 2, ...$$

Problem B.14: Consider 10 toses of a biased coin with probability of head equal to p.

(a) Find the probability that there are 3 heads in the first 5 tosses and 3 heads in the last 5 tosses.

(b) Given that there were a total of 6 heads heads find the probability that exactly 3 of them occurred in the first 5th trials.

(c) Find the probability that there are 5 heads in the first 8 tosses and 3 heads in the last 5 tosses.

Problem B.15: The joint density of two continuous random variables X and Y is given as follows

$$f\left(x,y\right)=c\left(x+y\right),0<x<2,\ 0<y<2$$

(a) What is the value of c?
(b) What is the marginal density of X?
(c) Find the best linear predictor of Y given $X=x$?

Problem B.16: The joint density of two continuous random variables X and Y is given as follows

$$f\left(x,y\right)=cx,\ 1<x<y<2$$

(a) What is the value of c?
(b) What is the marginal density of Y?
(c) Find the best linear predictor of Y given $X=x$?
(d) Determine the expected value of $1/X$ given that $Y=3/2$

Problem B.17: The amount X (in dollars) John takes to the casino each evening is a random variable with pdf

$$f_X\left(x\right)=cx,\ 0<x<400$$

At the end of each night, the amount Y that he has on leaving the casino is uniformly distributed between zero and twice the amount he took in.

(a) Determine the joint density for (X,Y)

(b) What is the probability that on any given night John makes apositive profit at the casino? Justify your reasoning.

(c) Find the pmf for Paul's profit on any particular night, $Z=Y-X$. What is $E[Z]$?

Problem B.17: Consider the communication of binary-valued messages over some transmission medium. Specifically, any message transmitted between locations is one of two possible symbols, 0 or 1. Each symbol occurs with equal probability. It is also known that any numerical value sent over this wire is subject to distortion; namely, if the value X is transmitted, the value Y received at the other end is described by Y = X + N where the random variable N represents additive noise that is independent of X. The noise N is normally distributed with mean $\mu=0$ and variance $\sigma^2=4$.

(a) Suppose the transmitter encodes the symbol 0 with the value $X=-2$ and the symbol 1 with the value $X=2$. At the other end, the received message is decoded according to the following rules:

• If $Y \geq 0$, then conclude the symbol 1 was sent.
• If $Y < 0$. then conclude the symbol 0 was sent.

Determine the probability of error for this encoding/decoding scheme. Reduce your calculations to a single numerical value.

(b) In an effort to reduce the probability of error, the following modifications are made. The transmitter encodes the symbols with a repeated scheme. The symbol 0 is encoded with the vector

$$\boldsymbol{X} = \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix}$$

and the symbol 1 is encoded with the vector

$$\boldsymbol{X} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

The vector

$$\boldsymbol{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} + \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix}$$

The vector

$$\boldsymbol{N} = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix}$$

represents the noise vector where each N_i is a random variable assumed to be normally distributed with mean $\mu = 0$ and variance $\sigma^2 = 4$. Assume each N_i is independent of each other and independent of the X_i's. Each component value of $\boldsymbol{Y}$ is decoded with the same rule as in part (a). The receiver then uses a majority rule to determine which symbol was sent. The receiver's decoding rules are:

• If 2 or more components of $\boldsymbol{Y}$ are greater than 0, then conclude the symbol 1 was sent.

• If 2 or more components of $\boldsymbol{Y}$ are less than 0, then conclude the symbol 0 was sent.

Determine the probability of error for this modified encoding/decoding scheme. Reduce your calculations to a single numerical value.