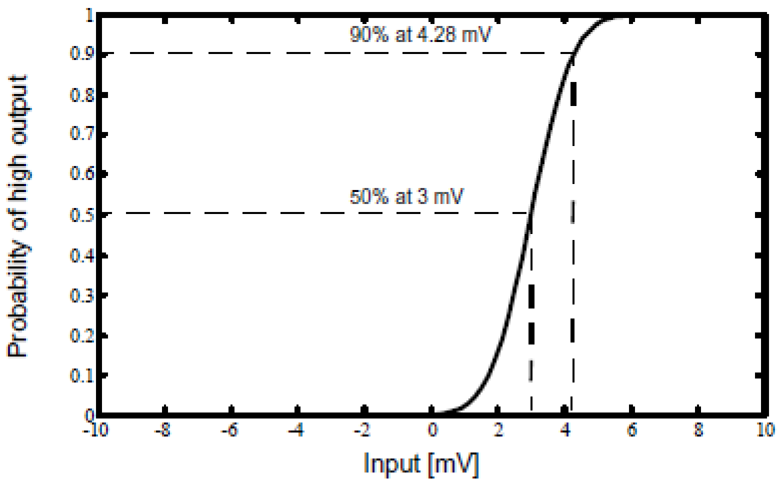
Background: It is said that comparators are the second most widely used electronic components after amplifiers. after "xxx" (Yang: what are the first most widely used?) A comparator is used to detect whether a signal (the signal could be differential, which means that the input is the difference between two signals and thus can be negative values) Yang: please, clarify this is greater or smaller than zero. The output of the comparator only has two states: 'high' and 'low'. For an ideal comparator, when the input signal is larger than zero, the output is 'high' and when the input signal is smaller than zero, the output is 'low'. However, in real life, a comparator has input offset and is subject to the effects of thermal noise, assumed to have a normal distribution. The input offset of a comparator is the input at which the output is equally likely to be high or low.

Problem: The output of a comparator is observed while slowly sweeping the input (a differential signal) over the range -10 mV to +10mV (Yang: how can the mV be negative? Please clarify). The fraction of time that the comparator is high is plotted as a function of the input voltage in following figure. (a) Estimate the comparator's input offset and noise. (b) For this comparator, what is the smallest input that can be resolved with an error rate of 10^{-12}?



Yang: it would be interesting (from the statistical point of view) to give more background about the way the above curve is obtained. For example, how many measurements are made to obtain each fraction of "high" status for the converter?

Solutions: (a) (the above figure is the CDF of a normal distribution) At an input of 3 mV, the comparator output is equally likely to be high or low; hence, the comparator's input offset is 3 mV. The 90% confidence interval for a normal distribution is at 1.28 times its SD (the rms (root mean square)).

$$\text{SD (rms) of noise} = (4.28-3)/1.28 \text{ mV} = 1 \text{ mV}$$

(b) For an error of 10^{-12} , the input must be away from the input offset 7 times SD of the noise. Hence, for a high output logic level the required input voltage is 3 mV+7(1 mV) =10 mV. For a low output logic level, the input must be below -4mV.

Reminder:

→ Assignment due Feb 10. code deadline

F 5:00 pm. Feb 10

→

→ Variable Transformation

→ DR Joint Distribution

Comparator

Small signal difficult to process

we wants to decide whether this Small signal is Larger or smaller than 0.

If $V_{in} > 0$, $V_{out} = '1'$;

$V_{in} < 0$, $V_{out} = '0'$.

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for an ideal comparator

Small Signal no matter how small it is

But in real life

① offset

For ideal ones, $V_{in} = 0$, it can't decide;

For real-life ones, ~~offset~~ V_{in} = offset, it can't deci

For example, offset = 3mV, $V_{in} < 3mV$, V_{out} = '0',

$V_{in} > 3mV$, V_{out} = '1'.

② Thermal noise

noise will follow a normal distribution.

Example 2. (Test a comparator)

A comparator, @ 4mV, 70% probability to output a logic 'high': @ 3.5mV, 60% probability to output a logic 'high'.

(a) What is the offset of the comparator?

(b) What is the SD (or rms) of the noise?

Solution: $qnorm(70\%) \rightarrow 0.524\ SD$

$qnorm(60\%) \rightarrow 0.253\ SD$

$(0.524 - 0.253)SD = (4 - 3.5)mV$

$SD = \frac{4 - 3.5}{0.524 - 0.253} mV =$

SD

Step 2: $\mu = 4mV - 0.524SD$

Random Variable Transformation

§ Given RV X with some distribution and $y = g(x)$, RV y is some function of x. Ask to find CDF and PDF or other characteristics of y.

Algorithm to resolve RVT

$\boxed{1.}$ Find the CDF of X (if not already given)

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(t)\,dt, \quad f_X(x) \to \text{pdf of } x$$

$\boxed{2}$ With the expression of y, substitude X with y in the CDF.

$\boxed{3}$ Solve the inequality inside the probability for x and plug it into $F_X(x)$.

$\boxed{4}$ Detertime the domain of $F_Y(g^{-1}(y))$.

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Tutorial B2

CDF given: $F(x) = 1 - \frac{1}{x^2}$, $x > 1$

$F(x) = 0$, for $x \leq 1$

(a) $Y = X^2$ pdf of Y?

using the algorithm for $y = x^2$

$\boxed{1}$ CDF of x given $F_X(x) = 1 - \frac{1}{x^2}$, $x > 1$

$\boxed{2}$ $F_Y(y) = P(Y \leq y) = P(x^2 \leq y)$ since $X > 0$

$\boxed{3}$ $F_Y(y) = P(x \leq \sqrt{y}) = 1 - \frac{1}{(\sqrt{y})^2}$

$= 1 - \frac{1}{y}$, $y > 1$.

pdf: $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(1 - \frac{1}{y}\right)$

$$= \begin{cases} \frac{1}{y^2}, & y > 1 \\ 0, & \text{otherwise} \end{cases}$$

(b) find pdf of $V = \frac{1}{x}$

Using the algorithm

$\boxed{1}$ $F_X(x) = 1 - \frac{1}{x^2}, \quad x > 1$

$\boxed{2}$ $F_V(v) = P(V \le v) = P(\underline{\frac{1}{X} \le v})$

$= 1 - (1 - \frac{1}{(1/v)^2}) = v^2, \quad 0 < v < 1$

Domain: $X > 1 \qquad 0 < V = \frac{1}{X} < 1.$

(c) $W = \ln(X)$

$\boxed{1}$ CDF of X: $F_X(x) = 1 - \frac{1}{x^2}, \quad x > 1.$

$\boxed{2}$ $F_W(w) = P(W \le w) = P(\ln x \le w)$

$= P(\underbrace{x \le e^w}_{\text{CDF of } x})$

$\boxed{3}$ $F_W(w) = 1 - e^{-2w}, \quad w > 0$

$= \cancel{1} = 1 - \frac{1}{(e^w}$

Domain: $x > 1. \quad w = \ln x > 0$

$f_W(w) = \frac{d}{dw} F_W(w) = 2e^{-2w}, \quad w > 0.$

DV Joint Distribution

Problem B.12

DRV, X, Y <u>joint pdf</u>

$$f(x,y) = c(x^2+y^2), \quad x = \{1, 2, 4\}$$
$$y = \{1, 3\}$$

(a) What is the value of c?

X \ Y	1	3
1	2c	10c
2	5c	13c
4	17c	25c

$$\sum_X \sum_Y P_{X,Y}(x,y) = 1$$ <u>(pmf sum is 1)</u>

$$c(2 + 10 + 5 + 13 + 17 + 25) = 1$$

$$c = \frac{1}{72}$$

(b) What is the prob. $P(Y < X)$?

$Y < X \Rightarrow (2,1), (4,1), (4,3)$

$P(Y<X) = P((2,1) \cup (4,1) \cup (4,3))$

$= 5c + 17c + 25c$

$= \frac{47}{72}$

(c) $P(Y > X) = 1 - P(Y \leq X)$

$= 1 - [P(Y<X) + P(Y=X)]$

$= 1 - \left[\frac{47}{72} + \frac{2}{72}\right]$

(f) Find marginal pmf $P_X(x)$ $P_Y(y)$?

$P_X(x) = \sum_{y} P_{X,Y}(x \geq y)$

$$P_Y(y) = \begin{cases} 48/72, & y = 3 \\ 24/72, & y = 1 \\ 0, & \text{else} \end{cases}$$

$\left.x = 1,\right\} = \sum_{y \in \{1,3\}} P_{X,Y}(1\,y)$

$= P(1,1) + P(1,3) = \frac{12}{72}.$

$\{x = 2,$
$x = 4,$

$$P_X(x) = \begin{cases} 12/72, & x = 1 \\ 18/72, & x = 2 \\ 42/72, & x = 4. \\ 0, & \text{else} \end{cases}$$

(g) $E(X)$, $E(Y)$, and $E(XY)$?

$$E(X) = \sum_x x \cdot P_X(x) = 1 \cdot \frac{12}{72} + 2 \cdot \frac{18}{72} + 4 \cdot \frac{42}{72}$$

$$= \frac{12 + 36 + 168}{72} = \frac{216}{72} = 3$$

$E(X) = 3$, $E(Y) = 7/3$

$\boxed{E(XY)}$? $E(g(X,Y)) = \sum_x \sum_y g(x,y) P(X=x, Y=y)$

$$E(XY) = \sum_x \sum_y x \cdot y \; p_{X,Y}(x,y)$$

$$= 1 \cdot 1 \, p(1,1) + \cdots$$

$$= \frac{484}{72}$$

Problem B.13

$$P(X=i \mid X+Y=n) = \frac{P(X=i \cap X+Y=n)}{P(X+Y=n)}$$

$$= \frac{P(X=i)P(Y=n-i)}{P(X+Y=n)} = \frac{(1-p)^{i-1}p \cdot (1-p)^{n-i-1}p}{P(X+Y=n)}$$

$X+Y=n$, for X or Y, X (or Y) ≥ 1.

② if we determine X, then Y is also determined.

for X there are $\binom{n-1}{1}$ choices.

$$= \frac{(1-p)^{i-1}p \cdot (1-p)^{n-i-1}p}{\cancel{P(X+Y=n)} \; C_{n-1}^{1}(1-p)^{n-1}p^2}$$

$$= \frac{1}{n-1}, \quad \text{for } i = 1, 2, \ldots$$

Or if we consider this problem in another way.

Given $X+Y=n$, $P(X=i \mid X+Y=n)$ will be prob. fix X @ i position divided by X has $(n-1)$ choices with equal prob. so $\frac{1}{n-1}$, for $i = 1, 2, \ldots$

Problem B.14

(a) $P(x=3) = \binom{5}{3} p^3 (1-p^2)$,

$P(Y=3) = \binom{5}{3} p^3 (1-p)^2$

~~P~~ Two independent events happen at the same time.

$$P((x=3) \cap (Y=3)) = P(x=3) P(Y=3)$$

$$= \binom{5}{3}^2 p^6 (1-p)^4 = 100 p^6 (1-p)^4$$

(b) Given : $x + Y = 6$ $\quad \underline{P(X=3 \mid X+Y=6)}$?

$$P(x=3 \mid x+Y=6) = \frac{P[(x=3) \cap (X+Y=6)]}{P(X+Y=6)}$$

$$= \frac{P(x=3) P(Y=3)}{P(X+Y=6)} = \frac{P(x=3) P(Y=3)}{P(x=1)P(Y=5) + P(x=2)P(Y=4) \cdots + \cdots P(x=5)P(Y=1)}$$

$$= \frac{100 p^6 (1-p)^4}{\binom{5}{1} p (1-p)^4 \binom{5}{5} p^5 + \cdots}$$

$$= \frac{100}{(5+50+100+50+5)} = \frac{10}{21}$$

B14(c)

× 10 positions $\boxed{1}\boxed{2}\boxed{3}\boxed{4}\boxed{5}$ $\boxed{6}\boxed{7}\boxed{8}$ $\boxed{9}\boxed{10}$

$\underbrace{\boxed{1}\boxed{2}\boxed{3}\boxed{4}\boxed{5}}_{\text{group 1}}$ $\underbrace{\boxed{6}\boxed{7}\boxed{8}}_{\text{group 2}}$ $\underbrace{\boxed{9}\boxed{10}}_{\text{group 3}}$

$\boxed{6}$-$\boxed{8}$ no heads: $\underline{P_1 = 0}$

$\boxed{6}$-$\boxed{8}$ 1 head: $P_2 = P(\boxed{1}\text{-}\boxed{5}\ 4\text{ heads})\,P(\boxed{6}\text{-}\boxed{8}\ 1\text{ head})\,P(\boxed{9}\boxed{10}\ 2\text{ heads})$

$$= \underbrace{\binom{5}{4}p^4(1-p)}_{g1}\ \underbrace{\binom{3}{1}p(1-p)^2}_{g2}\ \underbrace{\binom{2}{2}p^3}_{g3}$$

$$= \underline{15p^7(1-p)^3}$$

$\boxed{6}$-$\boxed{8}$ 2 heads $P_3 = P(\boxed{1}\text{-}\boxed{5}\ 3\text{ heads})\,P(\boxed{6}\text{-}\boxed{8}\ 2\text{ heads})\,P(\boxed{9}\boxed{10}\ 1\text{ head})$

$$= \underbrace{\binom{5}{3}p^3(1-p)^2}_{g1}\ \underbrace{\binom{3}{2}p^2(1-p)}_{g2}\ \underbrace{\binom{2}{1}p(1-p)}_{g3}$$

$$= \underline{60p^6(1-p)^4}$$

$\boxed{6}$-$\boxed{8}$ 3 heads $P_4 = P(\boxed{1}\text{-}\boxed{5}\ 2\text{ heads})\,P(\boxed{6}\text{-}\boxed{8}\ 3\text{ heads})\wedge P(\boxed{9}\boxed{10}\text{ no head})$

$$= \underbrace{\binom{5}{2}p^2(1-p)^3}_{g1}\ \underbrace{\binom{3}{3}p^3}_{g2}\ \underbrace{\binom{2}{0}(1-p)^2}_{g3} = \underline{10p^5(1-p)^5}$$

$P = P_1 + P_2 + P_3 + P_4$