Stat 547 C

Assignment 1

1. Consider a system of n antennae arranged in a linear order. Communication flows through the system provided no two consecutive antennae are down.

   (a) Suppose that $m < n$ antennae are down and the remaining $n - m$ are functional. How many linear orderings are there in which no pair of consecutive antennae are down?

   (b) Suppose that there are $n = 10$ antennae and the probability that m of them are down are as in the following table:

m	0	1	2	3	4	5 or more
Probability	0.35	0.39	0.19	0.06	0.01	0

   Calculate the probability that communication flows through this system.

2. Let $\{A_i : i \in I\}$ be a collection of sets. Prove "De Morgan's Law"

$$(\cup_i A_i)^c = \cap_i A_i^c$$

   and deduce that

$$(\cap_i A_i)^c = \cup_i A_i^c$$

3. Show that

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) \\ &\quad - P(A \cap B) + P(A \cap C) + P(B \cap C) \\ &\quad + P(A \cap B \cap C) \end{aligned}$$

More generally, use induction to prove that

$$P\left(\cup_{i=1}^{n} A_i\right) = \sum_{i_1=1}^{n} P\left(A_{i_1}\right) \qquad \text{(inclusion)}$$

$$- \sum_{i_1<i_2}^{n} P\left(A_{i_1} \cap A_{i_2}\right) \qquad \text{(exclusion)}$$

$$+ \sum_{i_1<i_2<i_3}^{n} P\left(A_{i_1} \cap A_{i_2} \cap A_{i_3}\right) \qquad \text{(inclusion)}$$

$$+ \cdots + (-1)^{r+1} \sum_{i_1<i_2<\cdots<i_r}^{n} P\left(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}\right)$$

$$+ \cdots + (-1)^{n+1} P\left(A_1 \cap \cdots \cap A_n\right)$$

$$= \sum_{r=1}^{n} (-1)^{r+1} \sum_{i_1<i_2<\cdots<i_r}^{n} P\left(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}\right)$$

4. You have n letters and n envelopes and randomly stuff the letters in the envelopes. What is the probability that at least one letter will match its intended envelope?

5. Two independent remote sensing devices, A and B, mounted on an airplane are used to detect and locate dead trees in a large area of forest land. The detectability of device A is 0.8 (that is, the probability that a dead tree will be detected by device A is 0.8), whereas the detectability of device B is 0.9. However, when a dead tree has been detected its location may not be pinpointed accurately by either device. Based on a detection from device A alone, the location of the dead tree can be accurately determined with probability 0.7, whereas the corresponding probability with device B alone is only 0.4. If a dead tree is detected by both devices, its location can be pinpointed with certainty. Determine the following.

   (a) The probability that a forest fire will be detected.

   (b) The probability that a forest fire will be detected by only one device

   (c) The probability of accurately locating a forest fire.

6. A, B and C take turns flipping a coin. The first one to get a head wins. Calculate the probability that A wins, that B wins and that C wins. Assume that A flips first, then B, then C, then A, and so on.

7. Suppose that items belong to one of two possible classes, $C = 0, 1$ (0="negative", 1="positive") . Suppose that some feature X is known (measured) for each item, but the class membership is unknown. Suppose that $X|C = 0 \sim f_0(x)$ and $X|C = 1 \sim f_1(x)$. That is, $f_i(x)$ are the conditional densities of X given $C = i$.

A classification function is a function $\hat{C}(x)$ that takes the values 0 and 1. When $\hat{C}(x) = 1$, the item is assigned to Class 1 and when $\hat{C}(x) = 0$, the item is assigned to Class 0. Define specificity and sensitivity of $\hat{C}(x)$ as follows:

$$Specificity\left(\widehat{C}\right) = P\left(\widehat{C} = 0|C = 0\right)$$

$$Sensitivity\left(\widehat{C}\right) = P\left(\widehat{C} = 1|C = 1\right)$$

(a) Show that

$$Specificity\left(\widehat{C}\right) = \int_{-\infty}^{\infty} \left[1 - \widehat{C}(x)\right] f_0(x)\, dx$$

$$Sensitivity\left(\widehat{C}\right) = \int_{-\infty}^{\infty} \widehat{C}(x) f_1(x)\, dx$$

(b) Let $0 < \alpha < 1$ be fixed. Consider the following classification function:

$$\widehat{C}_k(x) = 1 \iff f_1(x) > k f_0(x)$$

where k is chosen such that

$$Specificity\left(\widehat{C}_k\right) = P\left(\widehat{C}_k = 0|C = 0\right) = 1 - \alpha$$

Let $\hat{C}(x)$ be any other classifier with the same specificity, that is

$$\int \left[1 - \widehat{C}(x)\right] f_0(x)\, dx = 1 - \alpha,$$

Show that

$$Sensitivity\left(\widehat{C}_k\right) \geq Sensitivity\left(\widehat{C}\right)$$

and therefore, to find the optimal classifier, we can restrict attention to classification functions $\widehat{C}_k$.

(d) Suppose that $P(C = 0) = \pi_0$ and $P(C = 1) = 1 - \pi_0$. Calculate the missclassification error for $\widehat{C}_k$, that is, compute

$$Error\left(\widehat{C}_k\right) = P\left(\widehat{C}_k = 0, C = 1\right) + P\left(\widehat{C}_k = 1, C = 0\right)$$

(e) Suppose now that $f_0 = N(0, 1)$ and $f_1 = N(1, 1)$. What value of k minimizes the missclassification error?