Stat 547 C

Assignment 2

1. (a) Show that

   $$B = \limsup_n A_n = \{w : w \in A_n \text{ for infinite many values of } n\} = \{w : w \in A_n \ \text{ i.o.}\}$$

   and

   $$C = \liminf_n A_n = \{w : w \in A_n \text{ for all but finitely many values of } n\}.$$

   (b) Show that

   $$\left(\overline{\lim} A_n\right)^c = \underline{\lim} A_n^c$$

2. Suppose that a monkey seats in front of a computer and starts hammering keys randomly on the keyboard. Show that the famous Shakespeare play Hamlet will eventually appear (with probability 1) in its entirety on the screen, although our monkey is not particularly known for its good taste in literature. You can assume that the monkey picks characters uniformly at random on the keyboard, and that the successive key strokes are independent.

3. Let A_n be a sequence of independent events with

   $$\lim_{n\to\infty} P(A_n) = 0.$$

   Suppose that

   $$\sum^{\infty} P(A_n^c \cap A_{n+1}) < \infty$$

   Show that with probability one ultimately the events A_n cease to occur.

4. Suppose that $X_1, X_2, ..., X_n, ...$ are i.i.d. with common distribution F.

   Definition: The i$^{\text{th}}$ observation is a **record** if

   $$X_i > \max\{X_1, X_2, ..., X_{i-1}\}.$$

   Let

   $$A_i = \{X_i \ \text{ is a record}\} \qquad B_i = A_i^c = \{X_i \ \text{ is not a record}\}$$

   Show that:

(a) $P(A_i) = 1/i$

(b) $P(A_1 \cap A_2 \cap \cdots \cap A_i) = 1/i!$

(c) More generally, $P(C_1 \cap C_2 \cap \cdots \cap C_i) = P(C_1)\,P(C_2)\cdots P(C_i)$ where $C_j = A_i$ or $C_j = B_i$. Use this result to show that the events A_i are independent.

(d) Let R_n be the number of records in the sequence $X_1, X_2, ..., X_n$. Calculate $E(R_n)$ and $Var(R_n)$. Show that

$$\ln(n) + \frac{1}{n} \leq E(R_n) \leq \ln(n) + 1.$$

Find also an upper bound for $SD(R_n)$.

(e) Show that records will not cease to occur with probability 1.

(f) Show that consecutive records will cease to occur with probability 1.

5. Show that $E\{|X|\} - 1 \leq \sum_{n=1}^{\infty} P(|X| > n) \leq E\{|X|\}$.

6. Let $X_1, X_2, ...$ be a sequence of i.i.d. random variables.

   Show that

$$\text{(a)} \quad E|X_1| < \infty \Rightarrow P(|X_n| > n \quad \text{i.o.}) = 0,$$

   and

$$\text{(b)} \quad E|X_1| = \infty \Rightarrow P(|X_n| > n \quad \text{i.o.}) = 1.$$

7. Let $X_1, X_2, ...$ be a sequence of i.i.d. random variables. Show that

$$\text{(a)} \quad E|X_1| < \infty \Rightarrow \lim_{n\to\infty} \frac{X_n}{n} = 0 \quad \text{with probability 1}$$

   and

$$\text{(b)} \; E|X_1| = \infty \Rightarrow \limsup \frac{|X_n|}{n} = \infty \quad \text{with probability 1}$$

   Hint: Use the results in the previous problem and notice that $E\{|X|\} < \infty \Leftrightarrow E\{|cX|\} < \infty$, for all $c > 0$.

8. Suppose that X is a non-negative continuous random variable. Then

$$E(X) = \int_0^{\infty} P(X > x)\,dx = \int_0^{\infty} (1 - F(x))\,dx.$$

   Hint: Let $I(y \leq x) = 1$ if $y \leq x$ and equal to zero, otherwise. Then

$$x = \int_0^{\infty} I(y \leq x)\,dy.$$

   Hence,

$$E(X) = \int_0^{\infty} x f(x)\,dx = \int_0^{\infty} \left(\int_0^{\infty} I(y \leq x)\,dy \right) f(x)\,dx$$