Stat 547 C

Assignment 3

1. Suppose that $X \sim N\left(\mu, \sigma^2\right)$. Show that
$$\left(\frac{X-\mu}{\sigma}\right)^2 \sim Gamma\,(1/2, 1/2) = \chi^2_{(1)}$$

2. Suppose that $X_1, X_2, ..., X_n$ are i.i.d. $N\left(\mu, \sigma^2\right)$. Show that
$$\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2_{(n)}$$
Hint: Show first that if $X \sim N\left(\mu, \sigma^2\right)$ then
$$\left(\frac{X-\mu}{\sigma}\right)^2 \sim Gamma\,(1/2, 1/2) = \chi^2_{(1)}.$$
Now use the MGF's. Recall that the MGF for a Gamma r.v. is derived in the classnotes. You can use it.

3. Let $X_i \sim Poisson\,(\lambda_i)\ \ (i = 1, ..., k)$ be independent random variables, and let $N = \sum_{i=1}^{k} X_i$. Show that $\mathbf{X}|N = n \sim Multinomial(n, p_1, p_2, ..., p_k)$, with $p_i = \lambda_i / \sum_{i=1}^{k} \lambda_i$.

4. Show that (assuming that all the expectations and variances exist and are finite)
$$E\,(Y) = E\,(E\,[Y|X])$$
and
$$Var\,(Y) = E\,(Var\,[Y|X]) + Var\,(E\,[Y|X])\,.$$

5. Let $\{X_i\}_{i=0}^{\infty}$ and Y be independent random variables. Suppose that $X_i \sim Binomial\,(1, p)$ and $Y \sim Poisson\,(\lambda)$. Calculate the mean and variance of $S = \sum_{i=1}^{Y} X_i$.

6. Let X and Y be independent Gamma random variables with parameters (m, λ) and (n, λ), respectively. Show that
$$S = X + Y$$
$$R = \frac{X}{X+Y}$$

are independent, $S \sim Gamma\,(m+n,\lambda)$ and $R \sim Beta\,(m,n)$. **Note:** recall that the $Beta\,(m,n)$ density is

$$f\,(r;m,n) = \frac{\Gamma\,(m+n)}{\Gamma\,(m)\,\Gamma\,(n)} r^{m-1}\,(1-r)^{n-1}\,, \quad 0 < r < 1.$$

7. Let

$$f_{X,Y}\,(x,y) = \begin{cases} 1/\left(x^2y^2\right) & x \geq 1, y \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Compute the joint density of $U = XY$, $V = X/Y$

   (b) Compute the marginal densities of U and V

   (c) Compute the conditional density of $U|V$ and $V|U$.

8. Let

$$g\,(\theta) = E\,\{\rho\,(Y - \theta X)\}$$

   where $0 \leq \rho\,(t)$ is a bounded, continuously differentiable loss function. Suppose that X has a finite mean. Show that

$$g'\,(\theta) = -E\,\{\rho'\,(Y-\theta X)\,X\}\,.$$

   Note: Since $\rho\,(t)$ is a loss function, we have that $\rho\,(t)$ is non-decreasing on $[0,\infty)$ and non-increasing on $(-\infty, 0]$.

9. Suppose that $X \sim F$. Define, for all $0 < u < 1$,

$$F^{-1}\,(u) = \inf\,\{x : F\,(x) \geq u\}\,.$$

   Then,

(a) $F\left(F^{-1}\,(u)\right) \geq u$, with equality if F is continuous
(b) $F^{-1}\,(F\,(x)) \leq x$, with equality if F is increasing
(c) Show that

$$\left\{u : F^{-1}\,(u) \leq x\right\} = \{u : u \leq F\,(x)\}$$

and use this result to show that

$$F^{-1}\,(U) \sim F, \quad \text{where } \; U \sim Unif\,(0,1)\,.$$

(d) Show that

$$\{x : F\,(x) < u\} = \left\{x : x < F^{-1}\,(u)\right\}$$

(e) Use the results in (a) and (d) to show that, if F is continuous, then

$$F\,(X) \sim Unif\,(0,1)\,.$$