Stat 547 C
Assignment 4

Problem 1 *Let* $\mathbf{X} \sim N(\boldsymbol{\mu}, \Sigma)$ *with*

$$\boldsymbol{\mu} = \begin{pmatrix} 2 \\ 5 \\ 3 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \Sigma = \begin{pmatrix} 5 & 2 & 0 & 1 \\ 2 & 9 & 6 & 5 \\ 0 & 6 & 5 & 3 \\ 1 & 5 & 3 & 4 \end{pmatrix}$$

(a) *Find the joint distribution of*

$$\mathbf{Y} = \begin{pmatrix} X_1 - X_2 \\ X_3 - 1.5X_1 \\ X_4 - 2X_2 \end{pmatrix}$$

(b) *Find the joint conditional distribution of* $X_1 + X_2$ *and* $X_3 + X_4$ *given* $X_1 - X_3 = y$.

Problem 2 *Suppose that* $\mathbf{X} \in \mathbf{R}^3$ *is a multivariate normal vector with moment generating function*

$$M_{\mathbf{X}}(\mathbf{t}) = \exp\left\{3(t_1 + t_2) - 5t_3 + 2t_1^2 + 3\left(t_2^2 + t_3^2\right) - t_1 t_2 + t_3(t_2 - 2t_1)\right\}$$

(a) *Write down the joint density function for* $\mathbf{X}$. *Explain and justify your procedure.*

(b) *What is the joint conditional density function of* (X_1, X_2) *given* $X_3 = -4$?

(c) *Calculate* $P(1 < X_1 < 3, 1 < X_2 < 3)$. ***Hint:*** *You may wish to use the function "integrate" in R.*

Problem 3 *Let* $\mathbf{X}$ *be as in Problem 1.*

(a) *Find the function* $g^*(x_2, x_3, x_4)$ *that minimizes*

$$J(g) = E\{|X_1 - g(X_2, X_3, X_4)|\}$$

among all functions $g(x_2, x_3, x_4)$. *That is, find* g^* *such that*

$$E\{|X_1 - g^*(X_2, X_3, X_4)|\} \le E\{|X_1 - g(X_2, X_3, X_4)|\}, \quad \text{for all } g(x_2, x_3, x_4).$$

Hint: *first show that if* $X \sim N\left(\mu, \sigma^2\right)$, *then*

$$g(t) = E\{|X - t|\} > E\{|X - \mu|\}, \quad \text{for all } t \neq \mu.$$

(b) *Calculate the minimum value of* $J(g)$, *that is, calculate*

$$J(g^*) = E\{|X_1 - g^*(X_2, X_3, X_4)|\}.$$

Problem 4 *Suppose that* $\mathbf{X} = (X_1, X_2)'$ *has bivariate normal distribution with*

$$\begin{aligned} E(\mathbf{X}) &= \boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \\ cov(\mathbf{X}) &= \Sigma = \begin{pmatrix} \sigma_{11} & \rho\sqrt{\sigma_{11}\sigma_{22}} \\ \rho\sqrt{\sigma_{11}\sigma_{22}} & \sigma_{22} \end{pmatrix} \end{aligned}$$

Let

$$Y_i = \begin{cases} -1 & \text{if } X_i < \mu_i \\ 0 & \text{if } X_i = \mu_i \\ 1 & \text{if } X_i > \mu_i \end{cases}, \quad i = 1, 2$$

Define the "quadrant correlation", QC*, between* X_1 *and* X_2 *which is defined as the Pearson correlation between* Y_1 *and* Y_2 *:*

$$QC = \frac{E(Y_1 Y_2)}{\sqrt{E(Y_1^2)\, E(Y_2^2)}}.$$

Show that

(a) $QC = (2/\pi)arcsine(\rho)$

(b) $|QC| \leq |\rho|$.

Problem 5 ***(a)*** *Suppose that the vector* $\mathbf{X} = (X_1, X_2)'$ *has a "non-decreasing" bivariate normal distribution (that is* $0 < \rho < 1$*, where* ρ *is the correlation between* X_1 *and* X_2*). Show that*

$$\rho \geq corr[g(X_1), X_2]$$

for all functions $g(x_1)$.

Hint*: Show first that you can assume w.l.g. that* $E(X_1) = E(X_2) = E[g(X_1)] = 0$, $var(X_1) = var(X_2) = 1$ *and* $var[g(X_1)] = \rho^2$.

(b) *Similarly, show that if the distribution is non-increasing (that is,* $-1 < \rho < 0$*) then*

$$\rho \leq corr[g(X_1), X_2]$$

for all function $g(x_1)$.

Note: *(a) and (b) imply that the correlation between* X_1 *and* X_2 *cannot be "sharpened" (increased in absolute value) by making a transformation on* X_1.