Module 2: Sequences of Events

Ruben Zamar
Department of Statistics
UBC

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NESTED SEQUENCE OF EVENTS

Increasing Sequence of Events:

$$A_1 \subset A_2 \subset A_3 \subset \cdots \subset A_n \subset \cdots$$

In this case we say that the sequence A_n converges to $A = \cup_{i=1}^{\infty} A_i$.

In symbols:

$$\lim_{n \to \infty} A_n = \cup_{i=1}^{\infty} A_i = A$$

In short:

$$A_n \uparrow A.$$

NESTED SEQUENCE OF EVENTS

Decreasing Sequence of Events:

$$A_1 \supset A_2 \supset A_3 \supset \cdots \supset A_n \supset \cdots$$

In this case we also say that the sequence A_n converges to $A = \cap_{i=1}^{\infty} A_i$.

In symbols:

$$\lim_{n \to \infty} A_n = \cap_{i=1}^{\infty} A_i = A$$

In short:

$$A_n \downarrow A.$$

Continuity of P

If $A_n \uparrow A$ or $A_n \downarrow A$ then

$$\lim_{n \to \infty} P(A_n) = P(A)$$

Proof: Consider first the case $A_n \uparrow A$. Let $B_1 = A_1$ and for $i = 2, 3, \ldots$set

$$B_i = A_i \cap A_{i-1}^c = A_i \backslash A_{i-1}$$

The sets B_i are disjoint,

$$A_n = B_1 \cup B_2 \cup \cdots \cup B_n \quad \text{and} \quad A = \cup_{i=1}^{\infty} B_i$$

[draw a picture]. Then

$$\begin{aligned} P(A) &= P\left(\cup_{i=1}^{\infty} B_i\right) = \Sigma_{i=1}^{\infty} P(B_i) = \lim_{n\to\infty} \Sigma_{i=1}^{n} P(B_i) \\ &= \lim_{n\to\infty} P\left(\cup_{i=1}^{n} B_i\right) = \lim_{n\to\infty} P(A_n) \end{aligned}$$

Suppose now that $A_n \downarrow A$. The result follows from the previous case taking complement:

$$\begin{aligned}
A_1 &\supset A_2 \supset A_3 \supset \cdots \supset A \\
&\Rightarrow A_1^c \subset A_2^c \subset A_3^c \subset \cdots \subset A^c \\
&\Rightarrow \lim_{n\to\infty} A_n^c = A^c \\
&\Rightarrow P\left(\lim_{n\to\infty} A_n^c\right) = P\left(A^c\right) \\
&\Rightarrow \lim_{n\to\infty} P\left(A_n^c\right) = P\left(A^c\right) \quad \text{(by the previous result)} \\
&\Rightarrow \lim_{n\to\infty} \left[1 - P\left(A_n\right)\right] = 1 - P\left(A\right) \\
&\Rightarrow \lim_{n\to\infty} P\left(A_n\right) = P\left(A\right)
\end{aligned}$$

Example 1

$$
\begin{aligned}
A_n &= (0, 1 - \frac{1}{n}] \\
A_n &\uparrow A = (0, 1) \\
\lim_{n \to \infty} P\left((0, 1 - \frac{1}{n}]\right) &= P\left((0, 1)\right)
\end{aligned}
$$

Example 2

$$
\begin{aligned}
A_n &= (-\infty, x + \frac{1}{n}] \\
A_n &\downarrow A = (-\infty, x] \\
\lim_{n \to \infty} P\left((-\infty, x + \frac{1}{n}]\right) &= P\left((-\infty, x]\right)
\end{aligned}
$$

NESTED SEQUENCE OF EVENTS

We have seen that when $A_n \uparrow A$ or $A_n \downarrow A$ then

$$\lim_{n \to \infty} P(A_n) = P(A)$$

NON-NESTED SEQUENCE OF EVENTS

$$A_1, A_2, A_3, \cdots , \ A_n, \cdots$$

The sequence A_n may or may not converges to a limit set A.

Limit superior (lim-sup):

$$\overline{\lim}_{n\to\infty} A_n \quad = \quad \cap_{m=1}^{\infty} \cup_{n=m}^{\infty} A_n = \cap_{m=1}^{\infty} B_m$$

$$B_m \quad \downarrow \quad \overline{\lim}_{n\to\infty} A_n$$

Limit inferior (lim-inf)

$$\underline{\lim}_{n\to\infty} A_n \quad = \quad \cup_{m=1}^{\infty} \cap_{n=m}^{\infty} A_n = \cup_{m=1}^{\infty} C_m$$

$$C_m \quad \uparrow \quad \underline{\lim}_{n\to\infty} A_n$$

INTERPRETATION OF LIM-SUP

$$\overline{\lim}\ A_n \quad = \quad \cap_{m=1}^{\infty} \cup_{n=m}^{\infty} A_n$$

$$\overline{\lim}_{n\to\infty} A_n \quad = \quad \{w : w \in A_n \ \text{ i.o. } \} \quad , \qquad \text{i.o.} = \text{infinitely often}$$

Note: if the outcome $w \in \overline{\lim}\ A_n$, then A_n occurs infinitely often.

INTERPRETATION OF LIM-INF

$$\underline{\lim} A_n = \cup_{m=1}^{\infty} \cap_{n=m}^{\infty} A_n$$

$$\underline{\lim} A_n = \left\{ w : w \in A_n \text{ for all but a finite set of } n's \right\}$$

Note: if the outcome $w \in \underline{\lim} A_n$, then ultimately A_n always occurs

Clearly (specially from their meanings)

$$\underline{\lim} A_n \quad \subset \quad \overline{\lim} A_n$$

DEFINITION: If

$$\underline{\lim} A_n \quad = \quad \overline{\lim}\ A_n = A$$

then

$$\lim_{n \to \infty} A_n = A$$

In short

$$A_n \to A$$

CONTINUOUS FUNCTIONS AND CONVERGENT SEQUENCES OF NUMBERS

- Suppose that $f(x)$ is a continuous function and $\lim_{n\to\infty} x_n = x_0$. Then

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(x_0)$$

- "lim" can be taken **inside** the function.

CONTINUITY OF P

General Result: $A_n \rightarrow A \Rightarrow \lim_{n\rightarrow\infty} P(A_n) = P(A)$

In other words:

$$\lim_{n\rightarrow\infty} P(A_n) = P\left(\lim_{n\rightarrow\infty} A_n\right)$$

LIMIT INFERIOR AND LIMIT SUPERIOR

Let x_n be a sequence of real numbers.

Definition:

$$\liminf x_n = \underline{\lim} x_n = \sup_{m \geq 1} \inf_{n \geq m} x_n$$

and

$$\limsup x_n = \overline{\lim}\ x_n = \inf_{m \geq 1} \sup_{n \geq m} x_n$$

NOTE: $\underline{\lim}\ x_n$ and $\overline{\lim}\ x_n$ always exist (could be $\pm\infty$, though).

Result:

$$\inf x_n \leq \underline{\lim}\ x_n \leq \overline{\lim}\ x_n \leq \sup x_n$$

Proof: Let $\quad y_n = \inf_{k \geq n} x_k \qquad z_n = \sup_{k \geq n} x_k$

By definition, $y_n \uparrow$ and $z_n \downarrow$. So, for $n \leq m$ we have

$$y_1 \leq y_n \leq y_m \leq z_m \leq z_n \leq z_1$$

Therefore

$$\inf x_n \leq y_n \leq z_m \leq \sup x_n, \ \textbf{for all } m, n$$

Hence,

$$\inf x_n \leq \sup_{n \geq 1} y_n \leq \inf_{m \geq 1} z_m \leq \sup x_n$$

$$\inf x_n \leq \underline{\lim} x_n \leq \overline{\lim}\ x_n \leq \sup x_n$$

MORE PROPERTIES

- If $\overline{\lim}\ x_n = x$ then for all $\epsilon > 0$ there exist $n(\epsilon)$ such that $x_n < x + \epsilon$ for all $n \geq n_0(\epsilon)$
  In words: ultimately, the sequence is bounded above by $\overline{\lim}\ x_n + \epsilon$, for all $\epsilon > 0$

- If $\underline{\lim}\ x_n = x$ then for all $\epsilon > 0$ there exist $n(\epsilon)$ such that $x_n > x - \epsilon$ for all $n \geq n_1(\epsilon)$
  In words: ultimately, the sequence is bounded below by $\underline{\lim} x_n - \epsilon$, for all $\epsilon > 0$

- $\lim x_n = x$ if and only if $\underline{\lim} x_n = \overline{\lim} x_n = x$

  Proof: Exercise

Consider now the general case. Recall that by assumption

$$\underline{\lim}\ A_n = \overline{\lim}\ A_n$$

Therefore

$$P\left(\underline{\lim}\ A_n\right) = P\left(\overline{\lim}\ A_n\right)$$

Moreover

$$\begin{aligned} P\left(\underline{\lim}\ A_n\right) &= \lim_{m\to\infty} P\left(\cap_{n=m}^{\infty} A_n\right) \leq \underline{\lim} P\left(A_m\right) \\ &\leq \overline{\lim} P\left(A_m\right) \leq \lim_{m\to\infty} P\left(\cup_{n=m}^{\infty} A_n\right) = P\left(\overline{\lim}\ A_n\right) \end{aligned}$$

Hence

$$\underline{\lim} P\left(A_m\right) = \overline{\lim} P\left(A_m\right) = \lim_{m\to\infty} P\left(A_m\right)$$

Example

$$
\begin{aligned}
A_n &= (1 - \frac{1}{n}, 2 - \frac{1}{n}] \\
A_n &\longrightarrow A = [1, 2) \\
P\left((1 - \frac{1}{n}, 2 - \frac{1}{n}]\right) &\longrightarrow P\left([1, 2)\right)
\end{aligned}
$$

BOREL-CANTELLI

Let A_n be a sequence of events. We have the following results:

1. If $\sum_{i=1}^{\infty} P(A_n) < \infty$ then $P\left(\overline{\lim}\ A_n\right) = P(A_n \text{ i.o.}) = 0$

2. If $\sum_{i=1}^{\infty} P(A_n) = \infty$ **and the A_n are independent**

   then $P\left(\overline{\lim}\ A_n\right) = P(A_n \text{ i.o.}) = 1$

Example: Suppose that

$$X_n \sim Bin(1, p_n)$$

are independent Bernoulli random variables and

$$A_n = \{X_n = 1\} \Rightarrow P(A_n) = p_n.$$

Case 1: $p_n = 1/n^2$. Since $\sum_{i=1}^{\infty} 1/n^2 = \pi/6$, we have

$$P\left(A_n \;\text{ i.o.}\right) = 0.$$

All the possible sequences ultimately become an infinite string of zeros.

[Let $B_n = \{X_n = 0\}$, then $P\left(\underline{\lim} B_n\right) = 1$. Why?]

Case 2: $p_n = 1/n$. Since $\sum_{i=1}^{\infty} 1/n = \infty$ and **the events are independent** we have

$$P(A_n \text{ i.o.}) = 1.$$

The prob of "1" converges to zero. However, in all the possible sequences, "1" keeps reappearing.

[Then $P\left(\underline{\lim} B_n\right) = 0$. Why?]

PROOF OF BOREL-CANTELLI

1. Suppose that $\sum_{i=1}^{\infty} P(A_n) < \infty$. Then

$$P\left(\overline{\lim}\ A_n\right) = \lim_{n\to\infty} P\left(\cup_{m\geq n} A_m\right) \leq \lim_{n\to\infty} \sum_{m\geq n} P(A_m) = 0.$$

2. Suppose that $\sum_{i=1}^{\infty} P(A_n) = \infty$ **and the** A_n **are independent.** Then

$$\begin{aligned} 1 - P\left(\overline{\lim}\ A_n\right) &= P\left(\left[\overline{\lim}\ A_n\right]^c\right) \\ &= P\left(\underline{\lim} A_n^c\right) = \lim_{n\to\infty} P\left(\cap_{m\geq n} A_m^c\right) \end{aligned}$$

Now,

$$
\begin{aligned}
P\left(\cap_{m\geq n} A_m^c\right) &= \lim_{N\to\infty} P\left(\cap_{m=n}^{N} A_m^c\right) = \lim_{N\to\infty} \prod_{m=n}^{N} P\left(A_m^c\right) \\
&= \lim_{N\to\infty} \prod_{m=n}^{N} \left[1 - P\left(A_m\right)\right] \leq \lim_{n\to\infty} \prod_{m=n}^{N} \left[e^{-P(A_m)}\right] \\
&= \lim_{N\to\infty} \left[e^{-\sum_{m=n}^{N} P(A_m)}\right] = e^{-\infty} = 0.
\end{aligned}
$$

Hence,

$$
1 - P\left(\overline{\lim}\ A_n\right) \leq 0 \Rightarrow P\left(\overline{\lim}\ A_n\right) = 1
$$

EXAMPLE 2

Example: Suppose that $X_1, X_2, ..., X_n, ...$ are iid with common **continuous** distribution F.

The i$^{\text{th}}$ observation is a **record** if

$$X_i \quad > \quad \max\{X_1, X_2, ..., X_{i-1}\}.$$

Let

$$A_i = \{X_i \text{ is a record}\} \qquad B_i = A_i^c = \{X_i \text{ is not a record}\}$$

We have the following results:

(1) $P(A_i) = 1/i$ (why?)

(2) $P(A_1 \cap A_2 \cap \cdots \cap A_i) = P(X_1 < X_2 < \cdots < X_i) = 1/i!$ (why?)

(3) $P(C_1 \cap C_2 \cap \cdots \cap C_i) = P(C_1)\,P(C_2) \cdots P(C_i)$

where $C_j = A_i$ or $C_j = B_i$ (why?)

Hint:

$$P(B_i) \;=\; (i-1)/i \quad \text{and} \quad P(A_i) = 1/i.$$

On the other hand, there are exactly $i-1$ possible locations in the current sorted sequence for placing X_i if it is not a record and only a single possible position to place it if it is a record.

(4) Let $Y_i = I_{A_i}$. Then $Y_1, Y_2, ..., Y_n, ...$are independent Bernoulli r.v.'s with $p_i = 1/i$

(5) By Example 1, Case 2, we have $P\left(\overline{\lim} A_n\right) = 1$. Therefore, records will not cease to occur with probability 1.

(6) Let $D_i = A_i \cap A_{i+1}$ for $i = 1, 2, ...$ By independence,
$P\left(D_i\right) = 1/i(i+1)$.

By Example 1, Case 1, we have $P\left(\overline{\lim} A_n\right) = 0$. Therefore, eventually consecutive records will cease to occur with probability 1.