Characteristic Function

A complex valued random variable $X : \Omega \to \mathcal{C}$ is of the form

$$X(w) = Y(w) + iZ(w),$$

where Y and Z are real valued random variable. In this case, we define

$$E(X) = E(Y) + iE(Z)$$

provided

$$E(|Y|) < \infty \quad \text{and} \quad E(|Z|) < \infty.$$

Recall that

$$|X| = \sqrt{XX^*},$$

where

$$X^* = Y - iZ$$

is the conjugate of X. That is

$$|X| = \sqrt{[Y + iZ]\,[Y - iZ]}$$

$$= \sqrt{Y^2 - (iZ)^2} = \sqrt{Y^2 + Z^2}$$

Recall that

$$e^{ix} = \cos(x) + i\sin(x)$$

and so

$$\left|e^{ix}\right| = \sqrt{[\cos(x) + i\sin(x)]\,[\cos(x) + i\sin(x)]'}$$

$$= \sqrt{\cos^2(x) + \sin^2(x)}$$

$$= 1.$$

The characteristic function of a real valued random variable, X, is now defined as

$$\phi_X(t) = E\left(e^{itX}\right) = E\left\{\cos(tX)\right\} + iE\left\{\sin(tX)\right\}$$

Since

$$\left|e^{itX}\right|^2 = 1,$$

$\phi_X(t)$ is finite for all $t \in R$.

We will state (without proof) the following results:

Result 1:

(a)

$$\phi_{a+bX}(t) = e^{ita}\phi_X(bt)$$

(b) If $X_1, ..., X_n$ are independent then

$$\phi_{\sum_{j=1}^n X_i}(t) = \Pi_{j=1}^n \phi_{X_i}(t).$$

(c) If $E\left\{|X|^k\right\} < \infty$, for some $k \geq 1$, then

$$\phi_X(t) = \sum_{j=0}^k \frac{E\left(X^j\right)}{j!}(it)^j + o\left(t^k\right)$$

$$= 1 + itE(X) - \frac{t^2}{2}E\left(X^2\right) + \cdots + \frac{E\left(X^k\right)}{k!}(it)^k + o\left(t^k\right)$$

In particular, if $E(X) = 0$ and $Var(X) = \sigma^2 < \infty$, then

$$\phi_X(t) = 1 - \frac{\sigma^2}{2}t^2 + o\left(t^2\right).$$

(d) $\phi_X(t) = \phi_Y(t) \iff F_X = F_Y$

(e) **Levy's Continuity Theorem:** If $X_n \to_d X$ then $\phi_{X_n}(t) \to \phi_X(t)$.

Moreover, if $\phi_{X_n}(t) \to g(t)$, and $g(t)$ is **continuous at zero**, then $g(t)$ is the characteristic function of a random variable X and $X_n \to_d X$.

(f) If $Z \sim N(0,1)$, then $\phi_Z(t) = e^{-t^2/2}$.

Multivariate Case

The characteristic function of a real valued random vector, $\mathbf{X}$, is defined as follows:

$$\phi_{\mathbf{X}}(\mathbf{t}) = E\{\exp(i\mathbf{t}'\mathbf{X})\} = E\{\cos(\mathbf{t}'\mathbf{X})\} + iE\{\sin(\mathbf{t}'\mathbf{X})\}$$

Clearly $\phi_{\mathbf{X}}(\mathbf{t})$ is finite for all $\mathbf{t} \in R^p$.

We will state without proof the following result:

Result 2 (Multivariate Case):

(a)

$$E\left\{\exp\left(i\mathbf{t}'\left(\mathbf{a}+B\mathbf{X}\right)\right)\right\} = \exp\left(i\mathbf{t}'\mathbf{a}\right)E\left\{\exp\left(i\mathbf{t}'B\mathbf{X}\right)\right\}$$

$$= \exp\left(i\mathbf{t}'\mathbf{a}\right)E\left\{\exp\left(i\left(B'\mathbf{t}\right)'\mathbf{X}\right)\right\}$$

$$\phi_{\mathbf{a}+B\mathbf{X}}\left(t\right) = E\left\{\exp\left(i\mathbf{t}'\left(\mathbf{a}+B\mathbf{X}\right)\right)\right\}$$

$$= \exp\left(i\mathbf{t}'\mathbf{a}\right)E\left\{\exp\left(i\mathbf{t}'B\mathbf{X}\right)\right\}$$

$$= \exp\left(i\mathbf{t}'\mathbf{a}\right)E\left\{\exp\left(i\left(B'\mathbf{t}\right)'\mathbf{X}\right)\right\}$$

$$= \exp\left(i\mathbf{t}'\mathbf{a}\right)\phi_{\mathbf{X}}\left(B'\mathbf{t}\right)$$

(b) If $\mathbf{X}_1, ..., \mathbf{X}_n$ are independent then

$$\phi_{\sum_{j=1}^{n}\mathbf{X}_i}(\mathbf{t}) = \Pi_{j=1}^{n}\phi_{\mathbf{X}_i}(\mathbf{t}).$$

(c) If $E\left\{|X_l|^2\right\} < \infty$, for all $1 \leq l \leq p$, then

$$\phi_{\mathbf{X}}(\mathbf{t}) = 1 + iE(\mathbf{X})'(\mathbf{t}) - \frac{1}{2}\mathbf{t}'E(\mathbf{X}\mathbf{X}')\,\mathbf{t} + o\left(\|\mathbf{t}\|^2\right)$$

(d) $\phi_{\mathbf{X}}(\mathbf{t}) = \phi_{\mathbf{Y}}(\mathbf{t}) \Longleftrightarrow F_{\mathbf{X}} = F_{\mathbf{Y}}$

(e) **Levy's Continuity Theorem:** If $\mathbf{X}_n \to_d \mathbf{X}$ then $\phi_{\mathbf{X}_n}(\mathbf{t}) \to \phi_{\mathbf{X}}(\mathbf{t})$. Moreover, if $\phi_{\mathbf{X}_n}(\mathbf{t}) \to g(\mathbf{t})$, continuous at $\mathbf{0}$, then $g(\mathbf{t})$ is the characteristic function of a random vector $\mathbf{X}$ and $\mathbf{X}_n \to_d \mathbf{X}$.

(f) If $\mathbf{Z} \sim N(0, I)$, then $\phi_Z(t) = \exp\left(-\frac{1}{2}\mathbf{t}'\mathbf{t}\right)$.

The Central Limit Theorem (CLT)

Result 3: Suppose that $X_1, ..., X_n$ are iid with mean μ and finite variance σ^2. Then

$$\frac{\sqrt{n}\left(\bar{X}_n - \mu\right)}{\sigma} \rightarrow_d N\left(0, 1\right).$$

Proof: Let

$$Z_i = \frac{X_i - \mu}{\sigma}, \qquad i = 1, ..., n.$$

Then $Z_1, ..., Z_n$ are iid with mean 0 and finite variance 1 and (by Result 6 (c))

$$\phi_{Z_i}\left(t\right) = 1 - \frac{t^2}{2} + o\left(t^2\right).$$

Moreover,

$$\bar{Z} = \frac{\bar{X} - \mu}{\sigma}$$

and

$$\sqrt{n}\bar{Z} = \frac{\sqrt{n}\left(\bar{X} - \mu\right)}{\sigma}.$$

By Result 1 (a) and (b) we have

$$\phi_{\sqrt{n}\bar{Z}}(t) = \phi_{(1/\sqrt{n})\sum_{j=1}^{n} Z_i}(t)$$

$$= \Pi_{j=1}^{n} \phi_{Z_i}\left(t/\sqrt{n}\right)$$

$$= \left[1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right]^n \to e^{-t^2/2}.$$

Note: let

$$g_n(t) = \left[1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right]^n$$

Now,

$$\begin{aligned}
h_n(t) &= \log(g_n(t)) \\
&= n \log\left[1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right] \\
&= n\left[\log\left(1 - \frac{t^2}{2n}\right) + \frac{1}{1 - \frac{t^2}{2n} + \tilde{o}\left(\frac{t^2}{n}\right)} o\left(\frac{t^2}{n}\right)\right],
\end{aligned}$$

$$0 \leq \left|\tilde{o}\left(\frac{t^2}{n}\right)\right| \leq \left|o\left(\frac{t^2}{n}\right)\right|$$

$$\begin{aligned}
&= \log\left(1 - \frac{t^2}{2n}\right)^n + \frac{o\left(\frac{t^2}{n}\right) / \left(\frac{t^2}{n}\right)}{1 - \frac{t^2}{2n} + \tilde{o}\left(\frac{t^2}{n}\right)} t^2 \\
&\to \log\left(e^{-\frac{t^2}{2}}\right) = -\frac{t^2}{2}, \qquad \text{for all } t
\end{aligned}$$

Therefore

$$g_n(t) = \left[1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right]^n \to e^{-\frac{t^2}{2}}, \quad \text{for all } t$$

Multivariate CLT

Suppose that $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_n$ are iid with mean μ and covariance matrix Σ. Then

$$\sqrt{n}\left(\bar{\mathbf{X}}_n - \mu\right) \to_d N\left(\mathbf{0}, \Sigma\right).$$

Proof. Let $\mathbf{Y}_j = \mathbf{X}_j - \mu$, then

$$E\left(\mathbf{Y}_j\right) = \mathbf{0}, \;\; E\left(\mathbf{Y}_j \mathbf{Y}_j'\right) = \Sigma$$

Moreover,

$$\sqrt{n}\left(\bar{\mathbf{X}}_n - \mu\right) = \sqrt{n}\bar{\mathbf{Y}} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \mathbf{Y}_j$$

$$\phi_{\sqrt{n}\bar{\mathbf{Y}}}\left(\mathbf{t}\right) = \phi_{\sum_{j=1}^{n} \mathbf{Y}_j}\left(\frac{1}{\sqrt{n}}\mathbf{t}\right)$$

$$= \left[\phi_{\mathbf{Y}_1}\left(\frac{1}{\sqrt{n}}\mathbf{t}\right)\right]^n$$

$$= \left[1 - \frac{1}{2n}\mathbf{t}'\Sigma\mathbf{t} + o\left(\frac{\|\mathbf{t}\|^2}{n}\right)\right]^n$$

$$\to \exp\left\{-\frac{1}{2}\mathbf{t}'\Sigma\mathbf{t}\right\}$$

the Characteristic function of $N(0, \Sigma)$.

The Delta Method - Univariate Case

Suppose that

$$\sqrt{n}\left(X_n-\theta\right)\rightarrow_d N\left(0,\sigma^2\right)$$

Let $g\left(t\right)$ be a continuously differentiable function at θ. Then

$$\sqrt{n}\left(g\left(X_n\right)-g\left(\theta\right)\right)\rightarrow_d N\left(0,\left[g'\left(\theta\right)\right]^2\sigma^2\right)$$

Proof.

First of all we notice that by **Slutzky's Theorem (b)** we have that

$$X_n-\theta=\frac{1}{\sqrt{n}}\sqrt{n}\left(X_n-\theta\right)\rightarrow_p 0.$$

Moreover, by the Mean Value Theorem

$$g\left(X_n\right)=g\left(\theta\right)+g'\left(\theta_n\right)\left(X_n-\theta\right)$$

with θ_n between θ and X_n. Since $\left|\theta_n-\theta\right|\leq\left|X_n-\theta\right|$, we have that

$$\theta_n\rightarrow_p\theta$$

and since g' is continuous at θ,

$$g'(\theta_n) \to_p g'(\theta).$$

Therefore

$$\sqrt{n}\left(g(X_n) - g(\theta)\right) = g'(\theta_n)\sqrt{n}\left(X_n - \theta\right)$$

$$\to_d \quad g'(\theta)\, N\left(0, \sigma^2\right) = N\left(0, \left[g'(\theta)\right]^2 \sigma^2\right).$$

Example. Suppose that $Y_1, Y_2, ..., Y_n$ are i.i.d. $Exp(\lambda)$.

$$F_\lambda(y) = 1 - e^{-\lambda y}$$

$$E_\lambda(Y) = \frac{1}{\lambda}, \quad var(Y) = \frac{1}{\lambda^2}$$

Show that

(a) $\hat{\lambda} = 1/\bar{Y} \to_p \lambda,$

(b) $\sqrt{n}\left(\hat{\lambda} - \lambda\right) \to_d N\left(0, \lambda^2\right)$

(c) $\sqrt{n}\left(\hat{\lambda} - \lambda\right)/\hat{\lambda} \to_d N(0, 1)$

(d) Use (c) to construct and approximate 95% Confidence interval for λ if $n = 25$ and $\bar{y} = 20$.

Solution

(a) By the WLLN

$$\bar{Y} \to_p E(Y_1) = \frac{1}{\lambda}$$

By continuity of the function $g(t) = 1/t$, at $t = 1/\lambda \neq 0$,

$$\hat{\lambda} = \frac{1}{\bar{Y}} \to_p \lambda \tag{1}$$

(b) By the CLT

$$\sqrt{n}\left(\bar{Y} - 1/\lambda\right) \to_d N\left(0, 1/\lambda^2\right)$$

Let

$$g(t) = 1/t,$$

$$g'(t) = -1/t^2$$

$$g'(1/\lambda) = -\lambda^2$$

$$[g'(1/\lambda)]^2 = \lambda^4$$

By the delta-method, we get

$$\sqrt{n}\left(1/\bar{Y}-\lambda\right) \rightarrow_d N\left(0,\left(1/\lambda^2\right)\lambda^4\right) = N\left(0,\lambda^2\right) \qquad (2)$$

(c) By (1), (2) and Slutzky's Theorem,

$$\sqrt{n}\left(\hat{\lambda}-\lambda\right)/\hat{\lambda} \rightarrow_d N\left(0,\lambda^2\right)/\lambda = N(0,1) \qquad (3)$$

(d)

$$P\left(-1.96 \leq \sqrt{n}\left(\hat{\lambda}-\lambda\right)/\hat{\lambda} \leq 1.96\right) \approx 2\Phi(1.96)-1 = 0.95$$

$$\Rightarrow P\left(-\frac{1.96\hat{\lambda}}{\sqrt{n}}+\hat{\lambda} \leq \lambda \leq \hat{\lambda}+\frac{1.96\hat{\lambda}}{\sqrt{n}}\right) \approx 0.95$$

$$\Rightarrow \hat{\lambda}\left(1 \pm \frac{1.96}{\sqrt{n}}\right) \quad \text{is an approx 95\% CI for } \lambda$$

$$\Rightarrow \frac{1}{20}\left(1-\frac{1.96}{5}\right) \quad \text{is an approx 95\% CI for } \lambda$$

$$\Rightarrow (0.0304\ ,\ 0.0696) \ \text{ is an approx 95\% CI for } \lambda$$

The Delta-Method- Multivariate Case

Suppose now hat

$$\sqrt{n}\left(\mathbf{X}_n - \theta\right) \rightarrow_d N\left(\mathbf{0}, \Sigma\right)$$

Let $g\left(\mathbf{t}\right)$ be a continuously differentiable function at θ. Then

$$\sqrt{n}\left(g\left(\mathbf{X}_n\right) - g\left(\theta\right)\right) \rightarrow_d N\left(\mathbf{0}, \nabla_g\left(\theta\right)' \Sigma \nabla_g\left(\theta\right)\right)$$

where $\nabla_g\left(\mathbf{t}\right)$ is the gradient of g, that is,

$$\nabla_g\left(\mathbf{t}\right) = \left(\frac{\partial g\left(\mathbf{t}\right)}{\partial t_i}\right) = \begin{pmatrix} \frac{\partial g(\mathbf{t})}{\partial t_1} \\ \frac{\partial g(\mathbf{t})}{\partial t_2} \\ \vdots \\ \frac{\partial g(\mathbf{t})}{\partial t_p} \end{pmatrix}.$$

Proof:

By the Mean Value Theorem

$$g\left(\bar{\mathbf{X}}_n\right) = g\left(\theta\right) + \nabla_g\left(\theta_n\right)'\left(\mathbf{X}_n - \theta\right),$$

where

$$\theta_n = \left(1 - \alpha_n\right)\mathbf{X}_n + \alpha_n\theta,$$

for some $0 \leq \alpha_n \leq 1$. Therefore

$$\sqrt{n}\left[g\left(\bar{\mathbf{X}}_n\right) - g\left(\theta\right)\right] = \nabla_g\left(\theta_n\right)'\left[\sqrt{n}\left(\mathbf{X}_n - \theta\right)\right] \to \nabla_g\left(\theta\right)'\mathbf{Y},$$

with $\mathbf{Y} \sim N\left(\mathbf{0},\Sigma\right)$ (using the multivariate version of Slutzky's Theorem). Finally,

$$\nabla_g\left(\theta\right)'\mathbf{Y} \sim N\left(\mathbf{0}, \nabla_g\left(\theta\right)'\Sigma\nabla_g\left(\theta\right)\right).$$

The Lindeberg Condition

There is a more general version of the CLT that applies to **triangular arrays**:

$$X_{n,1}, X_{n,2}, \cdots, X_{n,k_n}$$

with $k_n \to \infty$ as $n \to \infty$. Here the $X_{n,j}$ $(j = 1, 2, ..., n_{k_n})$ are assumed independent, with mean 0 and variance

$$\sigma_{n,j}^2 = E\left(X_{n,j}^2\right), \qquad j = 1, 2, ..., k_n.$$

For example, if $k_n = n$,

$$\begin{array}{l} X_{1,1} \\ X_{2,1}, X_{2,2} \\ X_{3,1}, X_{3,2}, X_{3,3} \\ X_{4,1}, X_{4,2}, X_{4,3}, X_{4,4} \\ X_{5,1}, X_{5,2}, X_{5,3}, X_{5,4}, X_{5,5} \\ \cdots \end{array}$$

Let

$$S_n = \sum_{j=1}^{k_n} X_{n,j}$$

and

$$s_n^2 = \sum_{j=1}^{k_n} \sigma_{n,j}^2$$

Then

$$\frac{S_n}{s_n} = \frac{\sum_{j=1}^{k_n} X_{n,j}}{\sqrt{\sum_{j=1}^{k_n} \sigma_{n,j}^2}} \to_d N(0,1)$$

provided that for all

$$A_n = \frac{1}{s_n^2} \sum_{j=1}^{k_n} E\left(X_{n,j}^2 \quad I\left(X_{n,j}^2 > \epsilon s_n^2\right)\right) \to 0, \quad \text{for all } \epsilon > 0. \tag{4}$$

This is known as the **Lindeberg condition**. It can be shown that if

$$\frac{S_n}{s_n} \to_d N(0,1) \qquad \text{and} \quad \max_{1 \le j \le k_n} \frac{\sigma_{n,j}^2}{s_n^2} \to 0$$

then (4) holds.

The additional condition regarding the maximal ratio of variances going to zero is needed. Consider the following counter-example:

$$X_{n,j} = X_j \sim N\left(0, \sigma_j^2\right),$$

with

$$\sigma_1^2 = 1 \quad \text{and} \quad \sigma_n^2 = n s_{n-1}^2$$

Example 1: In the i.i.d. case Lindeberg condition is satisfied because

$$X_{n,j} = X_j, \quad \text{for all } n, j$$

$$Var\left(X_{n,j}\right) = \sigma^2 \quad \text{for all } n, j$$

$$s_n^2 = n\sigma^2$$

$$A_n = \frac{1}{n\sigma^2} \sum_{j=1}^{k_n} E\left(X_{,j}^2 \quad I\left(X_{,j}^2 > \epsilon n\sigma^2\right)\right)$$

$$= \frac{1}{\sigma^2} E\left(X_1^2 \quad I\left(X_1^2 > \epsilon n\sigma^2\right)\right) \to 0, \quad \text{as } n \to \infty,$$

by the DCT applied to

$$X_1^2 \quad I\left(X_1^2 > \epsilon n\sigma^2\right),$$

which is dominated by X_1^2 and converges to 0 as $n \to \infty$.

Example 2: The **simple linear regression model** provides and example where the “triangular array" version of the CLT is very useful.

Consider the model

$$Y_i = \alpha + \beta\left(x_i - \bar{x}\right) + U_i,$$

where $U_1, U_2, ..., U_n$ are independent, with mean zero and finite variance σ^2.

We will also assume that

$$\max_{1 \le i \le n} \frac{\left(x_i - \bar{x}\right)^2}{\sum\left(x_i - \bar{x}\right)^2} = b_n \to 0, \qquad \text{as } n \to \infty.$$

The Least Squares estimate of α and β are

$$\hat{\alpha} = \frac{\sum Y_i}{n} \qquad \text{and} \qquad \hat{\beta} = \frac{\sum \left(Y_i - \bar{Y}\right)(x_i - \bar{x})}{\sum (x_i - \bar{x})^2}$$

Notice that

$$\hat{\alpha} = \bar{Y} = \frac{\sum Y_i}{n}$$

$$= \frac{\sum \alpha + \beta (x_i - \bar{x}) + U_i}{n}$$

$$= \alpha + \frac{\sum U_i}{n},$$

Hence

$$E(\hat{\alpha}) = \alpha, \quad var(\hat{\alpha}) = \frac{\sigma^2}{n}$$

$$E\left(\hat{\beta}\right) = E\left(\frac{\sum\left(x_i - \bar{x}\right)E\left(Y_i - \bar{Y}\right)}{\sum\left(x_i - \bar{x}\right)^2}\right)$$

$$= \frac{1}{\sum\left(x_i - \bar{x}\right)^2}E\left(\sum\left(x_i - \bar{x}\right)Y_i\right)$$

$$= \frac{1}{\sum\left(x_i - \bar{x}\right)^2}\left(\sum\left(x_i - \bar{x}\right)E\left(\alpha + \beta\left(x_i - \bar{x}\right) + U_i\right)\right)$$

$$= \frac{\sum\left(x_i - \bar{x}\right)^2}{\sum\left(x_i - \bar{x}\right)^2}\beta = \beta$$

and

$$var\left(\hat{\beta}\right) = \frac{1}{\left[\sum\left(x_i - \bar{x}\right)^2\right]^2} \sum\left(x_i - \bar{x}\right)^2 var\left(Y_i\right)$$

$$= \frac{\sigma^2}{\sum\left(x_i - \bar{x}\right)^2}.$$

Now we will investigate the asymptotic distribution of $\hat{\beta}$.

$$\hat{\beta} = \frac{\sum \left(Y_i - \bar{Y}\right)(x_i - \bar{x})}{\sum (x_i - \bar{x})^2}$$

$$= \frac{\sum Y_i (x_i - \bar{x})}{\sum (x_i - \bar{x})^2}, \qquad \text{because } \sum \bar{Y} (x_i - \bar{x}) = 0$$

$$= \sum Y_i w_{n,i}$$

with

$$w_{n,i} = \frac{x_i - \bar{x}}{\sum (x_i - \bar{x})^2},$$

$$\sum w_{n,i} = 0,$$

$$\sum w_{n,i} x_i = \sum w_{n,i} (x_i - \bar{x})$$

$$= \frac{\sum (x_i - \bar{x})^2}{\sum (x_i - \bar{x})^2} = 1$$

$$\sum w_{n,i}^2 = \frac{1}{\sum (x_i - \bar{x})^2}$$

Moreover

$$\left(\hat{\beta} - \beta\right) = \left(\sum Y_i w_{n,i} - \beta\right)$$

$$= \left(\sum (\alpha + \beta x_i + U_i) w_{n,i} - \beta\right)$$

$$= \sum U_i w_{n,i}$$

Let

$$Z_{n,j} = U_j w_{n,j} = U_j w_j = Z_j, \qquad \text{for all } j, n$$

The subscript n is dropped from the notation for simplicity.

Then

$$E\left(Z_j\right) = 0, \;\; \sigma_j^2 = Var\left(Z_j\right) = w_j^2 \sigma^2,$$

and so

$$s_n^2 = \sum \sigma_j^2$$

$$= \sigma^2 \sum w_j^2$$

$$= \frac{\sigma^2}{\sum \left(x_i - \bar{x}\right)^2}$$

and

$$\frac{S_n}{s_n} = \frac{\sum_{j=1}^{n} Z_j}{s_n}$$

$$= \frac{\sqrt{\sum_{i=1}^{n} \left(x_i - \bar{x}\right)^2}}{\sigma} \sum_{j=1}^{n} U_j w_j$$

$$= \frac{\sqrt{\sum_{i=1}^{n} \left(x_i - \bar{x}\right)^2}}{\sigma} \left(\hat{\beta} - \beta\right)$$

Now, we check the Lindeberg Condition:

$$A_n = \frac{1}{s_n^2} \sum_{j=1}^{n} E\left(Z_j^2 \quad I\left(Z_j^2 > \epsilon s_n^2\right)\right)$$

$$= \frac{\sum (x_i - \bar{x})^2}{\sigma^2} \sum_{j=1}^{n} w_j^2 E\left(U_j^2 \quad I\left(w_j^2 U_j^2 > \epsilon s_n^2\right)\right)$$

$$= \frac{\sum (x_i - \bar{x})^2}{\sigma^2} \sum_{j=1}^{n} w_j^2 E\left(U_1^2 \quad I\left(w_j^2 U_1^2 > \epsilon s_n^2\right)\right)$$

$$\leq \frac{\sum (x_i - \bar{x})^2}{\sigma^2} \sum_{j=1}^{n} w_j^2 E\left(U_1^2 \quad I\left(U_1^2 > \epsilon \sigma^2 / b_n\right)\right)$$

$$= E\left(U_1^2 \quad I\left(U_1^2 > \epsilon \sigma^2 / b_n\right)\right) \frac{\sum (x_i - \bar{x})^2}{\sigma^2} \sum_{j=1}^{n} w_j^2$$

$$= \frac{E\left(U_1^2 \quad I\left(U_1^2 > \epsilon \sigma^2 / b_n\right)\right)}{\sigma^2} \to 0, \qquad \text{as } n \to \infty$$

Recall that

$$w_j^2 = \frac{(x_j - \bar{x})^2}{\left[\sum (x_i - \bar{x})^2\right]^2}$$

and

$$s_n^2 = \frac{\sigma^2}{\sum (x_i - \bar{x})^2}$$

Hence we have:

$$\frac{w_j^2}{s_n^2} = \frac{\sum (x_i - \bar{x})^2}{\sigma^2} \frac{(x_j - \bar{x})^2}{\left[\sum (x_i - \bar{x})^2\right]^2}$$

$$= \frac{1}{\sigma^2} \frac{(x_j - \bar{x})^2}{\sum (x_i - \bar{x})^2}$$

$$\leq \frac{1}{\sigma^2} b_n, \quad \text{for all } j$$

Hence

$$I\left(w_j^2 U_1^2 > \epsilon s_n^2\right) = I\left(\frac{w_j^2}{s_n^2} U_1^2 > \epsilon\right)$$

$$\leq I\left(b_n U_1^2 > \epsilon\right)$$

and therefore

$$A_n \leq \frac{\sum\left(x_i - \bar{x}\right)^2}{\sigma^2} \sum_{j=1}^{n} w_j^2 E\left(U_1^2 \quad I\left(b_n U_1^2 > \epsilon\right)\right)$$

$$= \frac{1}{\sigma^2} E\left(U_1^2 \quad I\left(b_n U_1^2 > \epsilon\right)\right) \frac{\sum\left(x_i - \bar{x}\right)^2}{\sum\left(x_i - \bar{x}\right)^2}$$

$$= \frac{E\left(U_1^2 \quad I\left(U_1^2 > \epsilon\sigma^2 / b_n\right)\right)}{\sigma^2} \rightarrow 0, \qquad \text{as } n \rightarrow \infty$$

by the DCT, because $b_n \rightarrow 0$, as $n \rightarrow \infty$.

Summary:

$$\frac{\sqrt{\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2}}{\sigma}\left(\hat{\beta}-\beta\right)\rightarrow_d N\left(0,1\right)$$

So, for large n

$$\frac{\sqrt{\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2}}{\sigma}\left(\hat{\beta}-\beta\right)\approx N\left(0,1\right)$$

$$\Rightarrow\hat{\beta}\approx N\left(\beta,\frac{\sigma^2}{\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2}\right)=N\left(E\left(\hat{\beta}\right),var\left(\hat{\beta}\right)\right)$$

Suppose now that

$$\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n\sigma^2} \to \frac{\sigma_x^2}{\sigma^2}$$

Then

$$\sqrt{n}\sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n\sigma^2}} \left(\hat{\beta} - \beta\right) \to_d N(0,1)$$

That is,

$$\sqrt{n}\left(\hat{\beta} - \beta\right) \to_d N\left(0, \frac{\sigma^2}{\sigma_x^2}\right).$$

Notice that

$$\frac{\sigma^2}{\sigma_x^2}$$

is a “noise to signal” ratio.